The Karol Adamiecki University of Economics
in Katowice

Volume 1 2004

Journal of
Economics &
Management

Jerzy Mika

ON THE SOLUTIONS OF GENERALIZED OPTIMIZATION PROBLEMS WITH THE LINEAR RESTRICTIVE CONSTRAINS

1. Optimization problems

The general form of optimization problems can be written in the following form:

$$\min\left\{ f(X) \;;\; X \in D \subseteq R^n \right\}$$

Furthermore, its optimal solution can be presented as follows:

$$X^* = \arg\ \min\left\{ f(X) \;;\; X \in D \subseteq R^n \right\}$$

In the theory of the optimization (compare [1]) optimization problems are specifically divided, according to the different kind inseparable and of mutually penetrating the classification.

Further considerations stated herein will concern the following the limited class of optimization problems: linear optimization problems and chosen types of the nonlinear optimization problems with the linear restrictive constrains. Some limitations based on the assumption that examined optimization problems will be nonrandom, onecriterial and statical and that sets of all decision variables of the examined optimization problems will be coherent. In that case examined linear optimization problems can be formulated in the so called canonical form:

$$\min\left\{ CX \;;\; AX = b \;,\; X \geq \Theta \right\}$$

Where:

A, b, C – matrices on facultative measurements, but consequential from the context.

Then the optimal solution of the examining problem is the vector:

$$X^* = \arg\ \min\left\{ CX \;;\; AX = b \;,\; X \geq \Theta \right\}$$

And the set of all optimal solutions is:

$$\{X^*\} = Arg\ \min\left\{ CX \;;\; AX = b \;,\; X \geq \Theta \right\}$$

where, due to the well-known fact of the equivalence of all forms of the linear programming problems it can be further restricted, without the loss of the generality limit to investigate only the chosen form of the linear optimization problems.

2. The problem of consistency and incosistency in the optimization procedures

The use of operation research methods in the management practice and control of economic or technological processes shows the inconsistency phenomenon in different phases of using of optimizations models is quite common. The naturalness of the occurrence of the inconsistency phenomenon of in such processes signals the classical rule of the rational activity ([4]). From formal perspective in cosistency of optimization models used in operation research application is only a simple opposition to the occurrence of the consistency and it means that the set of admissible solutions and optimal solutions of the examining optimization problem is empty and finally the problem solving is finished. Such approach is characteristic for the strictly mathematical area of the theory of the optimization. The different approach is allowed from practical point of view, when the focus is on the problem of the application of optimization problem and its solutions to the practical support of specific processes of optimal decision making. This leads to the problem of the elaboration specific and effective, but nonclassical tactics in cases of the inconsistency phenomenon used in optimization methods, and also of the adaptation of obtained results of these methods in optimal decision making processes. First of all the objective possibility of the occurrence of nonconsistent optimization models in optimal decision making processes demands specification of different feasible tactics strategies in such cases. That is why we assume that in the case of the inconsistency phenomenon in the use of optimization methods there are three possibilities of the conduct:

- resignation in the optimal decision making process from the suitable optimization problem when it proved to be inconsistent
- correction of the incosistent optimization problems used in the decision making process
- definition of accepted concept of generalized solutions of the incosistent optimization problems and how they can be soled and appliedin decision making processes.

The resignation is an extremely passive approach, which practically leads the procedure from the application of optimization methods in decision making processes to the initial stage, that is to the revision of initial stages of the methodology of operation research.

Correction of the incosistent optimization problems involves their suitable modification, as result of which corrected models will be close to the initial models and at the same time incosistent. Basic problem here is the acceptance of proper correction of parameters of incosistent problems.

The idea of generalized solutions used in incosistent optimization models in decision making processes demands proper definition of the concept of generalized solutions of optimization problems first and then preparation of effective methods of calculating then and accepting initial inconsistent models in suboptimal solutions. The concept of generalized solutions should be defined so, that sets of admissible and generalized optimal solutions for consistent optimization problems were identical with sets of possible and optimal solutions of these problems in the classical sense.

Next section of the paper focuses on the specification of generalized solutions for the chosen class of linear programing and related problems.

3. Generalized solutions of optimization problems

Let us assume that the problems of the optimization are linear canonical form, that is optimization problem with set of possible solutions of the following form:

$$D = \left\{ X \in R^n \ ; \ AX = b \ , \ X \geq \Theta \right\}$$

The set Δ of generalized solutions of the linear optimization problem is defined as follows:

$$\Delta = \left\{ \hat{X} \in R^n \ ; \ \left\| A\hat{X} - b \right\| = \min_{X \in R^n} \left\| AX - b \right\| \right\}$$

Consequently the set Δ of generalized solutions of the examining linear optimization problem can be presented in the following form:

$$\Delta = \left\{ \overline{X} \in R^n ; \sum_{j=1}^{n} \left(a_{ij} \overline{x}_j - b_i \right)^2 = \min_{X \in R^n} \left[\sum_{j=1}^{n} \left(a_{ij} x_j - b_i \right)^2 \right], \ X \geq \Theta \right\}$$

Besides we assume that possible generalized solutions of the linear optimization problem will be:

- each generalized solution defined with the above example with nonnegative values of all variables, if there are such vectors, or:
- the following arbitrarily defined vector:

$$\hat{X}^{(0)} = \left[\hat{x}_j^{(0)} \right]$$

where:

$$\hat{x}_j^{(0)} = \begin{cases} \hat{x}_j^0 & ; \ gdy: \quad \hat{x}_j^0 \rangle \ 0 \\ 0 & ; \ gdy: \quad \hat{x}_j^0 \leq 0 \end{cases}$$

if the set of generalized solutions Δ is the set one element, $\{\overset{\wedge 0}{x}\}$ consisting of $\overset{\wedge 0}{x}$, or:

- the zero vector – from the definition the trite solution trivial, if the set of generalized solutions Δ is not one element and it does not contain the vector of nonnegative components.

After the synthesis of the above-mentioned special cases we finally assume the following qualification of the set of possible solutions of general linear optimization problems of the researched canonical form:

$$\hat{D} = \begin{cases} \left\{ \hat{X} \in R^n ; \ \hat{X} \in \Delta , \ \hat{X} \geq \Theta \right\}; & when: \ R_+^n \cap \Delta \neq \varnothing \\ \hat{X}^{(0)} \quad ; & when: \ R_+^n \cap \Delta = \varnothing \ \ oraz \ \ \Delta = \left\{ \hat{X}^0 \right\} \\ \Theta \quad ; & when: R^n \cap \Delta = \varnothing \ \ or \ \ \Delta - include \ one \ element \end{cases}$$

Easy to notice that the set defined by the formula is always non-void, and it is in a special case which is also typical and essential especially in economic use of operation research, all elements of matrix A and of vector b, are not nectative and additionally the set is limited (compare [11]).

It is significant that the above qualification of possible generalized solutions of the linear optimization problem of the researched type is exhaustive in this meaning, that for any (both possible and impossible) optimization problem of the form, such defined generalized solutions always exist.

In compliance with recommended here idea of generalized solutions any optimization problems of linear canonical form are so inconsistent in this convention, that is for these optimization problems there are vectors, which are possible and optimal in the case of proposed idea of generalized solutions.

It shows also that the notion of generalized solutions defined with the above formula with its own range can embrace other, not only lineal optimization problems. The basic notion of generalized solutions can also be defined in a different way, using other than the Euclidean norm in the space R^m.

Particularly the set Δ_C of generalized solutions of the linear optimization problem according to Tschebyschev norm can be defined as follows:

$$\Delta_C = \left\{ \hat{X} \in R^n ;\ \left\| A\hat{X} - b \right\|_C = \min_{X \in R^n} \left\| AX - b \right\|_C \right\}$$

Consequently the set Δ_C of generalized solutions of the optimization problem according to Tschebyschev norm can be presented in the following form:

$$\Delta_C = \left\{ \overline{X} \in R^n ; \max_{1 \le i \le m} \left| \sum_{j=1}^{n} a_{ij}\overline{x}_j - b_j \right| = \min_{X \in R^n} \left[\max_{1 \le i \le m} \left| \sum_{j=1}^{n} a_{ij} x_j - b_i \right| \right],\ X \ge \Theta \right\}$$

The following set Δ_M of generalized solutions of the optimization problem can be defined according to the modular norm:

$$\Delta_M = \left\{ \hat{X} \in R^n ;\ \left\| A\hat{X} - b \right\|_M = \min_{X \in R^n} \left\| AX - b \right\|_M \right\}$$

Consequently the set Δ_C of generalized solutions of the optimization problem according to the modular norm can be presented in the following form:

$$\Delta_M = \left\{ \overline{X} \in R^n ; \sum_{i=1}^{m} \left| \sum_{j=1}^{n} a_{ij}\overline{x}_j - b_j \right| = \min_{X \in R^n} \left[\sum_{i=1}^{m} \left| \sum_{j=1}^{n} a_{ij}\overline{x}_j - b_j \right| \right],\ X \ge \Theta \right\}$$

In order to mark the above mentioned definite generalized possible and optimal solutions of sentences of the linear optimization of the researched type tools of the standard algebra methodology, so called generalized inverse matrices can be used.

4. The generalized inverse matrices method

The idea of the method of generalized inverse matrices in the linear optimization is based on ([10]) the application in the process to the research of optimal generalized solutions of problem of the linear optimization of the suitable type, special twoparametrics vector of the following form:

$$\mathbf{X(z,U) = P + Qz + KU}$$

Where, for the specific case of the optimization problem of the linear canonical form:

$$P = A^{+}b - \frac{M^{T}CA^{+}b}{\| M \|}$$

$$Q = \frac{M^{T}}{\| M \|}$$

$$K = E - A^{+}A - \frac{M^{T}M}{\| M \|}$$

$$M = C(E - A^{+}A)$$

A, b, c – parameters of the researched linear optimization problem,

A^{+} – generalized inverse matrix for the matrix A,

$z \in R$, $U \in R^{n}$ – the number and the vector, which are parameters of the vector X(z, U).

Though it is sensible and essentially well-founded there is the limitation of the above considerations to the situation, when sets of solutions or generalized solutions of researched optimization problems will be non-void and limited. Other cases have only exclusively theoretical meaning and show that the optimization model does not adequately reflect the decision-making process.

Concluding, the following theorem (the detailed proof is presented in [10]) of optimal solutions of the generalized linear optimization problems of researched form can be formulated.

Theorem

If the set of possible generalized solutions of the optimization problem of the linear canonical form is non-void and limited, the following vector:

$$\mathbf{X(z^*, U^*) = P + Qz^* + KU^*}$$

where:

$$\mathbf{z^* = min} \ \{ \ z \in R \ , \bigvee_{U \in R^n} \mathbf{X(z, U) \geq \Theta} \ \}$$

U^* – any vector from space R^n, for which the minimum is with the above formula

will be an optimal generalized solution of the given linear programming problem, and the number z^* will be an optimal value of the function criterion of this problem.

The general idea of generalized inverse matrices method in the linear optimization, of both searching for generalized optimal solutions of given optimization problems of the linear canonical form and wider class of optimization problems with the linear system of restrictive constrains, will include constructing the initial problem suitable for two – parametric vector X(z, U), and then determining such suitable values of parameters, that is number $z^* \in R$ and vector $U^* \in R^n$ so that vector $X(z^*, U^*)$ will be (in compliance with the above theorem) the optimal generalized solution, and the number z*, the value minimizing value of the function of the criterion of optimization problem of generalized possible solutions.

The basic version of the algorithm of the method of generalized inverse matrices, adapted to marking optimal solutions of generalized programming problems of the linear canonical form, can (compare [10]) be introduced in the form of the suitable three-stages procedure.

5. Generalizations and recommendations

A. Generality is the basic feature of presented idea of generalized solutions and generalized inverse matrices method of determining generalized optimal solutions of the linear optimization problems, especially that besides less essential specific cases (concerning potential practical use), they are effective for the linear optimization problems of examined types.

B. It is also important that the use of proposed idea of finding optimal solutions to generalized to solving consistent linear optimization problems of researched types always results in analogous results with these which would be obtained if classical concepts and calculation methods of the lineal optimization were used. Therefore proposed idea of generalized solutions may be used in linear optimization methods support to optimal decision making process regardless of whether (used in these decision processes) (the linear optimization problems) being consistent or inconsistent. While solving appropriate linear optimization problems (in the process of real decision problems) using presented here concept of generalized solutions, after optimal generalized solutions have been determined it should be checked if the result is a solution according to the classical theory of mathematical programming, if it is or a generalized solution in the researched approach.

C. It may not be sufficient to use linear models in some managerial, operational or optimal economic, decision-making processes. The nonlinearity of optimization models generally is a result of the so-called large scale effect or non additive in optimized processes. When nonlinear models of the mathematical programming problems, used in optimal decision-making processes, can be transformed into equivalent problems or linear optimization, problem sequence their generalized solutions may be determined, which then may be determined by a properly adapted variant of generalized inverse matrices method. These considerations can be generalized on optimization problems with the linear system of constraints of canonical form, formulated as follows:

$$\min\left\{ f(X) \; ; \; AX = b \; , \; X \geq \Theta \right\}$$

So-called problems of the convex optimization with a linear constrains are a special kind of this type of the nonlinear optimization. In this case additional assumptions are made of the salience and differentiability of the function of criterion in formulating of the above problem.

Bibligraphy

Czerwiński Z.: Matematyka na usługach ekonomii. PWN, Warszawa 1980.

Gass S.I.: Programowanie liniowe. PWN, Warszawa 1980.

Lange O.: Optymalne decyzje. PWN, Warszawa 1970.

Martos B.: Programowanie nieliniowe. Teoria i metody. PWN, Warszawa 1983.

Mika J.: O pewnej wersji metody macierzy pseudoodwrotnych w programowaniu liniowym. "Przegląd Statystyczny" 1982, nr 1.

Mika J.: Ob odnom metodie wyczislenia obobszczennych reszeni zadacz liniejnogo programirowania i jego primienieniach w planirowani Kurzfassungen der Beitrage, Halle-Wittenberg, Mathematik und Kybernetik in der Ökonomie VII, 1982.

Mika J.: On determining optimal generalized solution of different types of linear and nonlinear programming problems by means of pseudoinverse matrix method, Kurzfassungen der Beitrage, Magdeburg, Mathematik und Kybernetik in der Ökonomie VIII, 1985.

Mika J.: O wynikach eksperymentów obliczeniowych z zastosowaniem algorytmu metody macierzy pseudoodwrotnych w programowaniu liniowym. Prace Naukowe Akademii Ekonomicznej we Wrocławiu, Nr 413, 1989.

Mika J.: Uogólnione rozwiązania i korekcje sprzecznych modeli optymalizacyjnych we wspomaganiu procesów podejmowania decyzji ekonomicznych. Prace Naukowe Akademii Ekonomicznej w Katowicach, Katowice 1990.

Mika J.: Optymalne uogólnione rozwiązania zadań programowania matematycznego z liniowym układem warunków ograniczających w procesach wspomagania podejmowania decyzji. Prace naukowe Politechniki Śląskiej w Gliwicach, Organizacja i Zarządzanie, Z. 2, Gliwice 2000.

Nykowski I.: Programowanie liniowe. PWE, Warszawa 1980.