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DETECTING SEASONALITY VIA WAVELET METHODS

Summary: The multiresolution wavelet analysis is an effective tool that may be used to decompose an economic time series into its several natural components: a trend, business and seasonal cycles (of different frequencies), and a noise. The article provides a brief description of the theoretical model. The model is illustrated with its application to unemployment rate data. The empirical research carried out shows that cycles detected by wavelet filtering accurately reproduce the empirical series and its intrinsic properties.

Keywords: wavelet decomposition, detecting seasonality, unemployment rate.

Introduction

Some economic phenomena and their indices such as GDP, inflation, unemployment rate, energy usage etc. are featured by their intrinsic cyclic behaviour, that is why an analysed adequate time series should be properly decomposed into three components: trend, cycles (of one or more frequencies), and noise. Moreover, both the nature of an economic process and the empirical evidence show that such cycles may take place in time intervals of different length. This cycle length indeterminacy can yield some obstacles for an estimation procedure that is used to decompose an analysed time series.

Econometric models with periodicity, adaptive methods, time series models with seasonal adjustment, Fourier analysis of time series or other classical low-pass and high-pass filtering belong to a wide heterogeneous collection of models that serve to solve the question of cycles extraction while decomposing data. Such models usually do not capture cycle length indeterminacy or they even treat cycles as any other trend perturbations. One of the attempts that can be undertaken to avoid these disadvantages is usage of the wavelet time series decomposition. Since wavelets are both time and scale-localized, the wavelet analysis and provides better resolution in the time domain than classical methods.

Generally, an analysed time series $x(t)$ is to be decomposed via multiresolution wavelet analysis as: $x(t) = S(t) + \sum_{j=1}^{J} D_j(t)$, where $S(t)$ denotes cyclic component with periodicity greater than 2^J periods, and all $D_j(t)$'s denote cyclic components with periodicity between 2^{j-1} and 2^j periods. $S(t)$ represents a trend (*smooth*) of $x(t)$, while $D_j(t)$'s may represent business and seasonal cycles (*details*), except $D_1(t)$ which is of the highest frequency and represents a noise.

The main goal of the paper is to present the theoretical wavelet decomposition model (i.e. multiresolution analysis) and to describe its properties (section 2) having regard to detection of cycles. The described model is illustrated with an empirical study concerning unemployment rate quoted monthly by Central Statistical Office of Poland (section 3).

The paper is a continuation of the authors' research in the wavelet application field (see [Stachura, 2004, 2009] for instance), and the empirical research is inspired by [Yogo, 2008] treating an empirical study concerning the US GDP over the last 60 years. At first the paper was planned to carry comparative research treating Polish versus US GDP. Unfortunately, this idea failed since Polish free economy belongs to the young ones, so generally data older than 20 years may occur inappropriate and the macroeconomic indices quoted quarterly or annually since 1990 constitute time series that are not long enough. Thus, a monthly unemployment rate seems to be a good solution. Firstly, its quotation frequency guarantees a proper time series length. Secondly, among indices noted monthly the unemployment rate exhibits cyclic behaviour in different scales, because it has both seasonal oscillations as well as strong quasi-business-cycle fluctuations. Thirdly, the very unemployment seems to be a question of great social interest nowadays.

1. Discrete wavelet transform

1.1. Basic notions

Let $\boldsymbol{X} = (x_1, x_2, \ldots, x_M)^T$ be a vector of M records. For all $j \in \{1,\ldots, J\}$ (j designates number of so-called decomposition scale, and J is the decomposition scale order) and for all $k \in \{1, \ldots, 2^j\}$ wavelet coefficients of

X are defined as $W_{j,k} = \sum_{m=1}^{J} x_m \cdot \psi_{j,k}(\frac{m}{M})$, where $\psi_{j,k}$ are translated and scaled forms of a basic wavelet ψ (i.e. $\psi_{j,k}(s) = 2^{-j/2} \cdot \psi(2^{-j}s - k)$). ψ is chosen in such a way that $\{\psi_{j,k}\}$ constitute an orthonormal basis. Coefficients $W_{j,k}$ are brought together into vectors: $\mathbf{W}_j = (W_{j,1}, ..., W_{j,2^{J-j}})^T$ according to the same number of scale and then $\mathbf{W} = (\mathbf{W}_1^T, \mathbf{W}_2^T, ..., \mathbf{W}_J^T, \mathbf{V}_J^T)^T$ is the discrete wavelet transform of X.

In $\mathbf{W}$ there appears an additional properly constructed residual vector $\mathbf{V}_J = (V_1, ..., V_{M-2^J+1})^T$ that makes the correspondence between X and its transform $\mathbf{W}$ invertible (for details see [Percival, Walden, 2000] for example). Briefly speaking, we have an analogy to the discrete Fourier transform as long as in both situations any vector has its only transform and can be recovered from its transform, which means that two relations hold: $\mathbf{W} = \mathbf{\Psi} X$ and $X = \mathbf{\Psi}^T \mathbf{W}$ for some orthonormal matrix $\mathbf{\Psi}$ ($\mathbf{\Psi}^T \mathbf{\Psi} = \mathbf{I}$) determined by the choice of the basic wavelet ψ.

The discrete wavelet transform may be equivalently defined by a scaling filter $\{h_l\}_{l \in \{0,...,L-1\}}$ of length L and its conjugate filter $\{g_l\}_{l \in \{0,...,L-1\}}$ (where: $g_l = (-1)^{l+1} h_{L-l-1}$). This approach is very important for numerical applications and it does not require dyadic partition, which means that $M = 2^{J'}$ ($J' \geq J$) is no longer obligatory. Additionally, the appearance of the residual vector $\mathbf{V}_J$ is a straightforward consequence of the construction of vector $\mathbf{W}$, which is done via so-called recursive scheme (see: [Percival, Walden, 2000]) and works as follows. First, vector X is transformed by use of both filters $\{h_l\}$ and $\{g_l\}$ in order to get wavelet coefficients vector $\mathbf{W}_1$ and scaling coefficients vector $\mathbf{V}_1$ for scale 1. Then $\mathbf{V}_1$ is analogously transformed to get wavelet coefficients vector $\mathbf{W}_2$ and scaling coefficients vector $\mathbf{V}_2$ for scale 2, and so on.

Wavelets and filters of appropriately many zero moments N are noteworthy since they are very useful when eliminating non-stationarity from empirical data integrated of order $d > 0$. For every scale j, series $\mathbf{W}_j = \{W_{j,t}\}_t$ are stationary and zero mean processes as long as $N \geq [d + \frac{1}{2}]$. For instances in Daubchies filters case the number of zero moments is equal to the length L (see [Serroukh at al., 2000]). That is the reason why the wavelet analysis may be used straightforwardly for empirical data. One does not even need to detrend the data.

1.2. Multiresolution wavelet analysis

The mentioned relation $\boldsymbol{X} = \boldsymbol{\Psi}^T \mathbf{W}$ is usually called the *wavelet synthesis* of $\boldsymbol{X}$. Let us now rewrite this relation as follows:

$$\boldsymbol{X} = \boldsymbol{\Psi}^T \mathbf{W} = \boldsymbol{\Psi}_1^T \mathbf{W}_1 + \boldsymbol{\Psi}_2^T \mathbf{W}_2 + \ldots + \boldsymbol{\Psi}_J^T \mathbf{W}_J + \boldsymbol{\Phi}_J^T \mathbf{V}_J,$$

where the matrices $\boldsymbol{\Psi}_1, \boldsymbol{\Psi}_2, ..., \boldsymbol{\Psi}_J$ and $\boldsymbol{\Phi}_J$ are such submatrices of $\boldsymbol{\Psi}$ that are commensurate with the partition of $\mathbf{W}$ into its submatrices $\mathbf{W}_1$, $\mathbf{W}_2$, ..., $\mathbf{W}_J$, $\mathbf{V}_J$. Thus $\boldsymbol{\Psi}$ may be expressed as: $\boldsymbol{\Psi} = (\boldsymbol{\Psi}_1^T, \boldsymbol{\Psi}_2^T, \ldots, \boldsymbol{\Psi}_J^T, \boldsymbol{\Phi}_J^T)^T$. Such commensurate partitions of $\mathbf{W}$ and $\boldsymbol{\Psi}$ constitute the so called *multiresolution analysis* of $\boldsymbol{X}$ (see [Percival, Walden, 2000]). Namely, if we define $\mathbf{D}_j = \boldsymbol{\Psi}_j^T \mathbf{W}_j$ for all j and $\mathbf{S}_J = \boldsymbol{\Phi}_J^T \mathbf{V}_J$, then

$$\boldsymbol{X} = \mathbf{D}_1 + \mathbf{D}_2 + \ldots + \mathbf{D}_J + \mathbf{S}_J,$$

where each $\mathbf{D}_j$ represent the portion of the synthesis $\boldsymbol{X} = \boldsymbol{\Psi}^T \mathbf{W}$ attributable to scale j. $\mathbf{D}_j$'s are called *details* and the $\mathbf{S}_J$ is called *smooth* of $\boldsymbol{X}$. Noteworthy facts are that $\left\| \mathbf{D}_j \right\|^2 = \left\| \mathbf{W}_j \right\|^2$ for all j (since $\boldsymbol{\Psi}$ is orthonormal) and the sample variance of $\boldsymbol{X}$ is equal to $\frac{1}{M} \sum_j \left\| \mathbf{D}_j \right\|^2$.

2. Unemployment rate decomposition[1]

Throughout this section the multiresolution wavelet analysis is applied to study cyclic components of the monthly unemployment rate in Poland (256 records, Apr 1991 – Jul 2012)[2]. Figure 1 depicts the data that exhibits visible seasonal oscillations and strong quasi-business-cycle[3] fluctuations as well. The dashed vertical lines symbolize Januaries and the continuous vertical lines symbolize

[1] All computations and graphics are executed with use of R environment.

[2] A few first observations that were quoted in 1990 during the very first months after political changes in Poland were omitted. Number of the omitted data were chosen in such a way that the number of remaining data would be the highest as possible power of 2 (i.e. $256 = 2^8$).

[3] Such cycles are rather preferred to be called quasi-business-cycles as the unemployment rate is negatively correlated with GDP that forms the main determinant of business-cycles of an economy. Furthermore, extreme values of unemployment rate and of GDP are not expected to appear simultaneously.

months of extreme values of the data over a few years around (Jul 1994, Aug 1998, Feb 2003, Oct 2008, Feb 2012). Such lines are also depicted in the other figures inserted below.

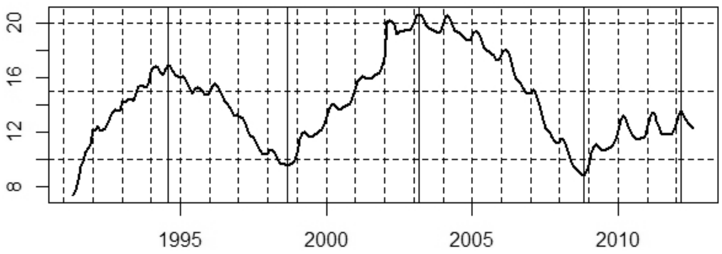

Fig. 1. Monthly unemployment rate

The empirical time series is decomposed via multiresolution wavelet analysis. The choice of basic wavelet is based on experimentation. First of all, basic filter must be sufficiently long to do away with improper artefacts in the wavelet transform. It occurs that filters that capture at least 12 adjacent observations work well in practice and there is no substantial improvement when lengthening filters more than that, which is not surprising as the empirical series exhibits strong annual seasonality. Secondly, among filters provided by R environment (Daubechies, Least Asymmetric, Best Localized, Coiflet) the most popular Daubechies filter was chosen because no substantial differences emerged among decompositions done with use of Daubechies filter and the other ones. And thirdly, one technical issue should be noted. The multiresolution wavelet analysis is applied in its so called reflective version, which means that the empirical series length is doubled by attaching observations in the reversed order with respect to time. This procedure allows to omit undesirable effects of overestimation or underestimation of the end values of the empirical time series smooth.

Therefore, the decomposition $x(t) = S_4(t) + D_4(t) + D_3(t) + D_2(t) + D_1(t)$ is illustrated in Figures 2-6, for Daubechies filter of length 12 and the decomposition order equal to 4. Since $D_j(t)$'s denote cyclic components with periodicity between 2^{j-1} and 2^j months, it is obvious that $D_4(t)$, $D_3(t)$, $D_2(t)$, $D_1(t)$ reveal components for about 2 year, annual, 6 month and quarter oscillations respectively.[4]

[4] These scales of oscillations motivate the prior selection of $J = 4$.

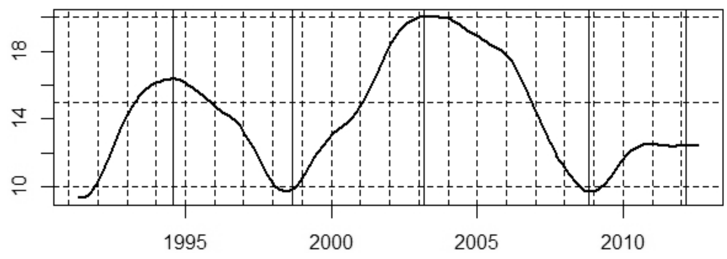

Fig. 2. Unemployment rate *smooth* (S_4)

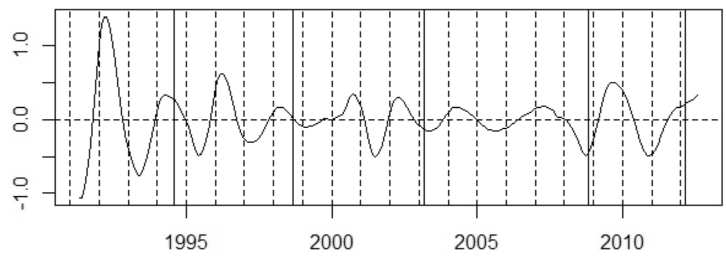

Fig. 3. 2 year oscillations *detail* of unemployment rate (D_4)

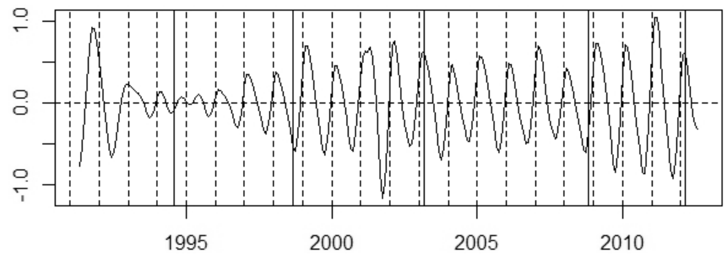

Fig. 4. Annual oscillations *detail* of unemployment rate (D_3)

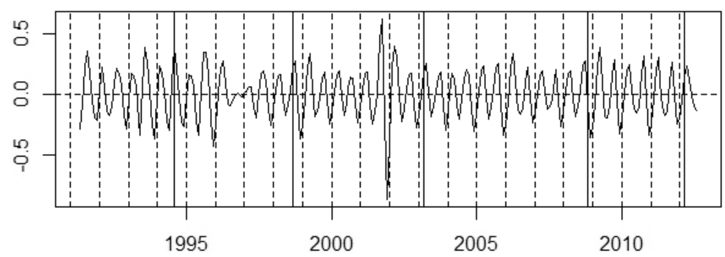

Fig. 5. 6 month oscillations *detail* of unemployment rate (D_2)

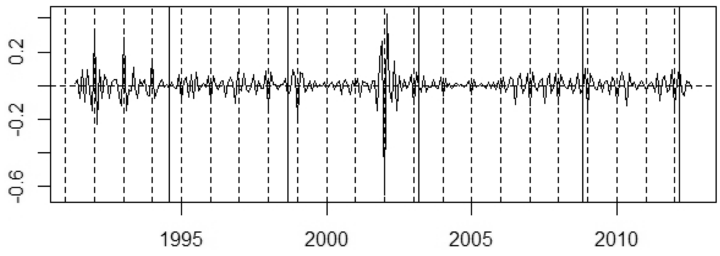

Fig. 6. Quarter oscillations *detail* of unemployment rate (D_1)

Casting an eye over the Figures 1-6 exhibits a very good adjustment of the S_4 series to the empirical data (it is confirmed by Pearson, Spearman, and Kendall correlation values between the empirical series and its smooth, these values equal to 0. 9835, 0. 9815, and 0.8914 respectively). Also a remarkable match of the details (except the noise D_1) to the cyclic behaviour of the employment rate can be noticed. Moreover, the sum of details $D_4(t)$, $D_3(t)$, $D_2(t)$ is fully accordant with the empirical series, since the local minima and maxima of the sum not only portray seasonal cyclic properties of the empirical series added to the smooth representing cyclic-type trend, but also 4 of the minima and maxima coincide almost perfectly with 4 characteristic extreme values of the empirical series as well (see Figure 7).

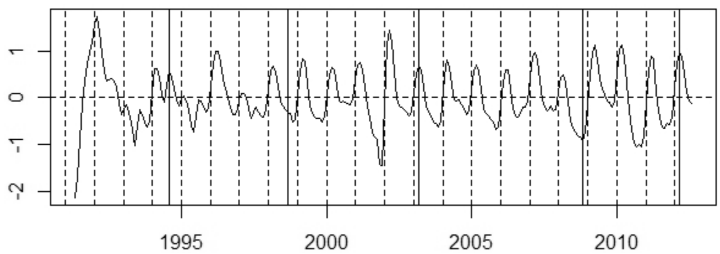

Fig. 7. Sum of the *details* ($D_4 + D_3 + D_2$)

Figures 1-7 show also a specific advantage of wavelet decomposition over the Fourier analysis or the classical seasonal adjustment. Namely, all of the details, and their sums as well, are not periodic functions though they detect periodical components of the empirical time series. Moreover, especially the detail of scale 4 and the smooth portray changeable duration of the trend cycles for proper time scales, which is not the case for example in Fourier analysis as it is based on sine and cosine cycles with determined cycle lengths. In that sense the analysis of the empirical time series discloses two quasi-business-cycles of different duration, which represent a slowly oscillating trend (first cycle is spanned between May 1991 and May 1998 including 7 years and 1 months with maximum in Jul 1994 and the other is spanned between May 1998 and Feb 2009 including 10 years and 9 months with maximum in Apr 2003). The mentioned oscillating trend is modified by the sum of details $D_4(t)$, $D_3(t)$, $D_2(t)$ representing oscillations of the data at higher frequencies.

Let us state that in such circumstances the current changes in the unemployment rate (Jul 2010 – Jul 2012) are caused by short cycle effects rather than by the slowly oscillating trend behaviour (compare right ends of the plots depicted in Figures 2 and 7). Nevertheless, long-time unemployment rate cycle seems to be at its plateau at the moment, which makes this component unpredictable, provided that the detected trend is not a cycle in a rigorous sense.

Conclusions

The presentation of the theoretical model and an example of its empirical application have clearly demonstrated some special features of the multiresolution wavelet analysis. Among specific features the most important ones, which seem to be advantages over other methods, are the following.

- There is no need to pre-analyse data and detrend time series with use of additional methods, since the wavelet transform may be successfully applied straightforwardly to unmodified data. What is more, one can even eliminate a non-stationarity problem this way.
- Wavelet analysis detects trends of any type. For instance, typical linear-type trends (research by [Yogo, 2008]) or cyclic-type ones (the hereby presented research) may be detected.
- A time series may be decomposed into several components with respect to different frequencies: trend (of different types), cycles (of different frequencies), and noise. Each of such component depicts oscillations of empirical data at certain scales and component variations sum up to the total variation of empirical series.
- As the wavelets are both time and scale-localized the mentioned components, reflecting periodic changes of data at certain scales, have their oscillations perturbed both by unstable cycle duration as well as by unstable amplitude value, while moving on time axis.

- If such a decomposition is done properly, it may suggest future behaviour of an analysed time series with respect to proper frequencies. And the decomposition can be used as an auxiliary tool to forecast a time series.

Concluding, it should be also noted that wavelet analysis is a very convenient, quick and effective tool in many cases, particularly due to the fact that a lot of ready-made computational packs are available. Nevertheless, wavelet analysis should be employed having regard to its other specific features that, to some extent, may be treated as weak points. Among them, naive and multidimensional geometrical approach should be mentioned, which implies that any result obtained with use of wavelet analysis must be subjected to careful confirmation in terms of some theory accordance with a described reality or in terms of a common sense at least.

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WYKRYWANIE SEZONOWOŚCI Z WYKORZYSTANIEM METOD FALKOWYCH

Streszczenie: Wieloskalowa analiza falkowa jest efektywnym narzędziem, które z powodzeniem można stosować do dekompozycji ekonomicznych szeregów czasowych na takie składowe, jak trend, cykl koniunkturalny, cykle sezonowe (różnych skal) oraz szum. W opracowaniu przedstawiono krótki opis modelu teoretycznego analizy wieloskalowej, który następnie zilustrowano na podstawie danych rzeczywistych dotyczących stopy bezrobocia. Przeprowadzone badania empiryczne pokazują, że filtry falkowe we właściwym stopniu odtwarzają szereg empiryczny oraz jego swoiste własności.

Słowa kluczowe: analiza falkowa, wykrywanie sezonowości, stopa bezrobocia.