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ON MOMENTS OF QUADRATIC FORM IN NORMAL RANDOM VARIABLES

1. Moments of the quadratic form

Let the random column vector **Y** has m-dimensional and nonsingular normal distribution. Its expected value is the zero vector and its variance covariance matrix is denoted by **Σ**. Hence **Y** ~ N(**μ**, **Σ**). Let **B** is a symetric non-random matrix of degree m. Let us consider the following quadratic form:

$$Q = \mathbf{Y}^T \mathbf{B} \mathbf{Y} \qquad (1.1)$$

Mathai and Provost (1992), p. 53, proved that

$$E(Q)^r = \left\{ \sum_{r_1=0}^{r-1} \binom{r-1}{r_1} d^{(r-1-r_1)} \sum_{r_2=0}^{r_1-1} \binom{r_1-1}{r_2} d^{(r_1-1-r_2)} \ldots \right\}$$

where:

$$d^{(k)} = 2^k k! \left\{ tr(\mathbf{B\Sigma})^{k+1} + (k+1)\boldsymbol{\mu}^T (\mathbf{B\Sigma})^k \mathbf{B}\boldsymbol{\mu} \right\}, \quad k = 0, 1, 2, \ldots$$

$$r_{i+1} \le r_i - 1 \text{ for i=0,1,...,r and } r_0=1$$

Particularly:

$$E(Q) = \sum_{r_1=0}^{0} \binom{0}{0} d^{(0)} = d^{(0)} = tr\,(\mathbf{B\Sigma}) + \boldsymbol{\mu}^T \mathbf{B}\, \boldsymbol{\mu}$$

$$E(Q)^2 =$$

$$\sum_{r_1=0}^{1} \binom{1}{r_1} d^{(1-r_1)} \sum_{r_2=0}^{r_1-1} \binom{r_1-1}{r_2} d^{(r_1-1-r_2)} = \binom{1}{0} d^{(1)} + \binom{1}{1} d^{(0)} \binom{0}{0} d^{(0)} =$$

$$= d^{(1)} + \left(d^{(0)}\right)^2 = 2\, tr\,(\mathbf{B\Sigma})^2 + 4\, \boldsymbol{\mu}^T (\mathbf{B}\, \boldsymbol{\Sigma})\, \mathbf{B}\, \boldsymbol{\mu} + (\, tr\,(\mathbf{B\Sigma}) + \boldsymbol{\mu}^T \mathbf{B}\, \boldsymbol{\mu})^2$$

$$D^2(Q) = d^{(1)} = 2\, tr\,(\mathbf{B\Sigma})^2 + 2\, \boldsymbol{\mu}^T \mathbf{B}\, \boldsymbol{\Sigma}\, \mathbf{B}\, \boldsymbol{\mu}$$

Now, we are going to derive an another representation of the moments of the quadratic form Q under the assumptions that the vector **Y** has m-dimensional and nonsingular normal distribution N (**O, Σ**).

The moment generating function of the random variable Q is (see: Mathai and Provost (1992), p. 41):

$$M(t) = [u(t)]^{-\frac{1}{2}} \tag{1.2}$$

where:

$$u(t) = \det(\mathbf{I} - 2t\,\mathbf{B}\,\mathbf{\Sigma}) \tag{1.3}$$

The n-th moment of the random variable Q is determined by the equation:

$$E(Q^n) = M^{(n)}(0),\ \mathbf{n} = 1, 2, \dots \tag{1.4}$$

where $M^{(\mathbf{n})}(t)$ is n-th derivative of the function M(t). Let $u^{(n)} = u^{(n)}(t)$ be the n-th derivative of the function u(t). The first derivative of the generating moments function is:

$$M^{(1)}(t) = v_1(t)\, u^{(1)}(t) = v_1 u^{(1)} \tag{1.5}$$

where:

$$v_1(t) = v_1 = v_1^{(0)} = -\frac{1}{2} u^{-\frac{3}{2}} \tag{1.6}$$

Let us define the following function:

$$v_e(t) = v_e = (-1)^e 2^{-e} \left[\prod_{h=0}^{e-1} (2h+1) \right] u^{-(2e+1)/2} \tag{1.7}$$

The moment generating function (derived in the appendix 3.1) is as follows:

$$\begin{cases} M^{(1)} = v_1 u^{(1)} \\ M^{(n)} = v_1 u^{(n)} + \sum_{k=2}^{n} v_k \left\{ \prod_{h=0}^{k-2} \left[\sum_{i_h = k-h-1}^{i_{h-1}-1} \binom{i_{h-1}-1}{i_h} u^{(i_{h-1}-i_h)} \right] \right\} u^{(i_{k-2})}, \quad n \geq 2 \end{cases} \tag{1.8}$$

where: $i_{-1} = n$ and $i_o = i$.

Particularly:

$$M^{(2)} = v_1 u^{(2)} + v_2 \left[u^{(1)}\right]^2$$

$$M^{(3)} = v_1 u^{(3)} + 3 v_2 u^{(2)} u^{(1)} + v_3 \left[u^{(1)}\right]^3$$

Let us consider the following determinant:

$$u(t) = \det(\mathbf{I} + \mathbf{C}) \quad (1.9)$$

where:

$$\mathbf{C} = -2\,t\,\mathbf{G} \tag{1.10}$$

The matrix **G** is of degree m and t is real value.

The theorem 3.1 leads to the following result:

$$u(t) = 1 + \sum_{h=0}^{m-1} \sum_{\{k_1,\dots,k_h\}}^{\binom{m}{n}} c(k_1,\dots,k_h) \tag{1.11}$$

where $c(k_1, \dots, k_h)$ is the principal minor of the matrix **C**. It is obtained as the determinant of the matrix formed by removing the rows and columns, identified by the indexes $\{k_1, \dots, k_h\}$ from the original matrix **C.** Similarly, let us denote the principal minor of the matrix **G** by $g(k_1, \dots, k_h)$. Particularly, if $h = 0$ then $c = \det(\mathbf{C})$ and $g = \det(\mathbf{G})$. If h=1,

$$\sum_{\{k_1,\dots,k_{m-1}\}} g(k_1,\dots,k_{m-1}) = \operatorname{tr}\mathbf{G}$$

The expression (1.11) can be rewritten in the following way:

$$u(t) = 1 + \sum_{h=0}^{m-1} (-2)^{m-h} t^{m-h} p_h \tag{1.12}$$

where:

$$p_h = \sum_{\{k_1,\dots,k_h\}}^{\binom{m}{h}} g(k_1,\dots,k_h) \tag{1.13}$$

The r-th derivative of the function u(t) is as follows:

$$u^{(r)}(t) = \sum_{h=0}^{m-r} (-2)^{m-h} (m-h)(m-h-1)\dots(m-h-r+1) t^{m-h-r} p_h, \quad r \le m \tag{1.14}$$

or

$$u^{(r)}(t) = \sum_{h=0}^{m-r} (-2)^{m-h} \frac{\Gamma(m-h+1)}{\Gamma(m-h-r+1)} t^{m-h-r} p_h, \quad r \le m \tag{1.15}$$

The equation (1.7) can be rewritten as follows:

$$u^{(r)}(t) = \sum_{h=0}^{m-r-1} (-2)^{m-h} \frac{\Gamma(m-h+1)}{\Gamma(-h-r+1)} t^{m-h-r} p_h + (-2)^r \Gamma(r+1) p_{m-r} \quad (1.16)$$

where p_h is explained by the expression (1.13), where $g(k_1, ..., k_h)$ is the principal minor of the matrix **G = B Σ** obtained by removing the rows and columns identified by the sequence of labels $\{k_1, ..., k_h\}$ from the matrix **G**. Hence:

$$u^{(r)}(0) = \begin{cases} (-2)^r r! p_{m-r} & \text{for} \quad r = 1, 2, ..., m \\ 0 \text{ for } r > m \end{cases} \quad (1.17)$$

Particularly:

$$u^{(1)}(0) = -2p_{m-1} = -2\mathrm{tr}G$$

$$u^{(2)}(0) = 8p_{m-2} = 8 \sum_{\{k_1,...,k_{m-2}\}}^{\binom{m}{m-2}} g(k_1,...,k_{m-2})$$

$$u^{(3)}(0) = -48p_{m-3}$$

The expression (1.7) leads to the following one because u(0)=1:

$$v_e(0) = (-1)^e 2^{-e} \prod_{h=0}^{e-1} (2h+1) \quad (1.18)$$

Particularly:

$$v_1(0) = -\frac{1}{2}, \quad v_2(0) = \frac{1}{2}\frac{3}{2}, \quad v_3(0) = -\frac{1}{2}\frac{3}{2}\frac{5}{2}, \quad v_4(0) = \frac{1}{2}\frac{3}{2}\frac{5}{2}\frac{9}{2}$$

The expression (1.4), (1.8), (1.13), (1.17) and (1.18) lead to the following one:

$$E(Q) = p_{m-1} = \sum_{\{k_1,...,k_{m-1}\}} g(k_1,...,k_{m-1}) = \sum_{i=1}^{m} g_{ii} = \mathrm{tr}[g_{ij}] = \mathrm{tr}\,\mathbf{G}$$

Hence:

$$E(Q) = p_{m-1} = \mathrm{tr}\,(\mathbf{B\Sigma}) \quad (1.19)$$

For n > 1

$$E\left(Q^n\right) = (-2)^{n-1}\Gamma(n+1)p_{m-n} +$$

$$+\sum_{k=2}^{n}(-2)^{-k}\left(\prod_{h=0}^{k-1}(2h+1)\right)\left\{\prod_{h=0}^{k-2}\left[\sum_{i_h=k-h-1}^{i_{h-1}-1}\binom{i_{h-1}-1}{i_h}(-2)^{(i_{h-1}-i_h)}\cdot\right.\right.$$

$$\left.\left.\cdot\Gamma\left(i_{h-1}-i_h+1\right)p_{m-i_{h-1}+i_h}\right]\right\}(-2)^{i_{k-2}}\Gamma\left(i_{k-2}+1\right)p_{m-i_{k-2}} \qquad (1.20)$$

where: i_{-1}=n, i_o=i and p_{m-n}=0 for m-n<0.

Particularly:

$$E\left(Q^2\right) = -4p_{m-2} + 3p_{m-1}^2 \qquad (1.21)$$

$$D^2(Q) = 2p_{m-1}^2 - 4p_{m-2} \qquad (1.23)$$

Let us note that:

$$D^2(Q) = 2p_{m-1}^2 - 4p_{m-2} = 2\left(\sum_{i=1}^{m} g_{ii}\right)^2 - 4\sum_{i=1}^{m}\sum_{\substack{j=1\\j>i}}^{m}\left(g_{ii}g_{jj} - g_{ij}g_{ji}\right) =$$

$$= 2\sum_{i=1}^{m} g_{ii}^2 + 4\sum_{i=1}^{m}\sum_{\substack{j=1\\j>i}}^{m} g_{ii}g_{jj} - 4\sum_{i=1}^{m}\sum_{\substack{j=1\\j>i}}^{m} g_{ii}g_{jj} + 4\sum_{i=1}^{m}\sum_{\substack{j=1\\j>i}}^{m} g_{ij}g_{ji} = 2\sum_{i=1}^{m}\sum_{j=1}^{m} g_{ij}g_{ji} =$$

$$= 2\mathrm{tr}\left[g_{ij}\right]^2 = 2\mathrm{tr}\mathbf{G}^2 = 2\mathrm{tr}\left(\mathbf{B\Sigma}\right)^2$$

$$E\left(Q^3\right) = (-2)^2\Gamma(4)p_{m-3} + \frac{3}{4}\sum_{i=1}^{2}\binom{2}{i}(-2)^{(3-i)}\Gamma(4-i)p_{m-3+i}(-2)^i\,\Gamma(i+1)p_{m-i} +$$

$$-\frac{3\cdot 5}{2^3}\binom{2}{2}(-2)^{(3-2)}\Gamma(4-2)p_{m-3+2}\binom{1}{1}(-2)^{(2-1)}\,\Gamma(2-1+1)p_{m-2+1}(-2)^1\Gamma(1+1)p_{m-1} =$$

$$= 24\,p_{m-3} + \frac{3}{4}\left(2\cdot 4\cdot 2\,p_{m-2}(-2)\,p_{m-1} + (-2)\,p_{m-1}4\cdot 2\,p_{m-2}\right) +$$

$$-\frac{15}{8}(-2)\,p_{m-1}(-2)\,p_{m-1}(-2)\,p_{m-1}$$

$$E\left(Q^3\right) = 24\,p_{m-3} - 36\,p_{m-2}p_{m-1} + 15\,p_{m-1}^3 \qquad (1.23)$$

$$\eta_3(Q) = E(Q - E(Q))^3 = 8\left(3p_{m-3} - 3p_{m-2}p_{m-1} + p_{m-1}^3\right) \qquad (1.24)$$

Hence, the expressions (1.19)–(1.24) show new representation of the moments of the quadratic form Q.

2. Application

The obtained results can be applied to approximation of the distribution function of Q. For instance, this distribution can be approximated by means of series expansions or as well as by method of Pearson's curves.

Review of application of distribution properties of quadratic forms and their functions are presented e.g. by Magnus (1986, 1990), Mathai and Provost (1992) or Domański, Pruska and Wagner (1998). They consider some applications involving distribution of quadratic form in problems connected with chi-square goodness-of-fit tests, ratio of two quadratic forms. Moreover, some econometric applications are possible, too.

Let $Q_1 = \mathbf{Y}^T \mathbf{B}_1 \mathbf{Y}$ and $Q_2 = \mathbf{Y}^T \mathbf{B}_1 \mathbf{Y}$. The distribution of the ratio:

$$R = \frac{Q_1}{Q_2} \tag{2.1}$$

can be transformed in the following way

$$F(r) = P\{R < r\} \tag{2.2}$$

$$F(r) = P\{Q_1 < rQ_2\}$$

$$F(r) = P\{Q_1 - rQ_2 < 0\}$$

$$F(r) = P\{\mathbf{Y}^T \mathbf{B}(r)\, \mathbf{Y} < 0\} \tag{2.3}$$

where:

$$\mathbf{B}(r) = \mathbf{B}_1 - r\mathbf{B}_2 \tag{2.4}$$

Hence, expressions (2.2) – (2.3) show that the distribution of the ratio R is determined by the distribution of the quadratic form $Q(r) = \mathbf{Y}^T \mathbf{B}(r)\, \mathbf{Y}$. Mathai and Provost (1992) show several particular cases of the distribution Q(r). Wywiał (1995) considered ratio of two quadratic forms Q_1, Q_2 measuring the ex-post errors of two predictors. Hence, the obtained results allow it compare the r accuracy of the predictors.

3. Appendixes

Appendix 3.1.

On the basis of the expression (1.5) and the well known Leibnitz's theorem (see e.g. Fichtenholz (1980)) we have:

$$M^{(n)}(t)=\sum_{i=0}^{n-1}\binom{n-1}{i}u^{(n-i)}v_1^{(i)} \tag{3.1}$$

This and the expression (1.7) let us show that:

$$v_{e-1}^{(1)}=v_e u^{(1)} \tag{3.2}$$

The moment generating function can be derived on the basis expressions (1.5)--(1.7), (3.1) and (3.2) in the following way:

$$M^{(n)}=u^{(n)}v_1+\sum_{i=1}^{n-1}\binom{n-1}{i}u^{(n-1)}v_1^{(i)}=u^{(n)}v_1+\sum_{i=1}^{n-1}\binom{n-1}{i}u^{(n-i)}\left(u^{(1)}v_2\right)^{(i-1)}=$$

$$=v_1u^{(n)}+v_2\binom{n-1}{1}u^{(n-1)}u^{(1)}+\sum_{i=2}^{n-1}\binom{n-1}{i}u^{(n-i)}\sum_{i_1=0}^{i-1}\binom{i-1}{i_1}u^{(i-i_1)}v_2^{(i_1)}=$$

$$=v_1u^{(n)}+v_2\binom{n-1}{1}u^{(n-1)}u^{(1)}+\sum_{i=2}^{n-1}\binom{n-1}{i}u^{(n-i)}u^{(i)}v_2+$$

$$+\sum_{i=2}^{n-1}\binom{n-1}{i}u^{(n-i)}\sum_{i_1=1}^{i-1}\binom{i-1}{i_1}u^{(i-i_1)}v_2^{(i_1)}=$$

$$=v_1u^{(n)}+v_2\sum_{i=1}^{n-1}\binom{n-1}{i}u^{(n-i)}u^{(i)}+\sum_{i=2}^{n-1}\binom{i-1}{i}u^{(n-i)}\sum_{i_1=1}^{i-1}\binom{i-1}{i_1}u^{(i-i_1)}\left(v^{(1)}v_3\right)^{i_1-1}=$$

$$=v_1u^{(n)}+v_2\sum_{i=1}^{n-1}\binom{n-1}{i}u^{(n-1)}u^{(i)}+\sum_{i=2}^{n-1}\binom{n-1}{i}u^{(n-i)}\binom{i-1}{1}u^{(i-1)}u^{(1)}v_3+$$

$$+\sum_{i=3}^{n-1}\binom{n-1}{i}u^{(n-i)}\sum_{i_1=2}^{i-1}\binom{i-1}{i_1}u^{(i-i_1)}\sum_{i_2=0}^{i_1=1}\binom{i_1-1}{i_2}u^{(i_1-i_2)}v_3^{(i_2)}=$$

$$= v_1 u^{(n)} + v_2 \sum_{i=1}^{n-1} \binom{n-1}{i} u^{(n-i)} u^{(i)} + v_3 \sum_{i=2}^{n-1} \binom{n-1}{i} u^{(n-i)} \binom{i-1}{1} u^{(i-1)} u^{(1)} +$$

$$+ v_3 \sum_{i=3}^{n-1} \binom{n-1}{i} u^{(n-i)} \sum_{i_1=2}^{i-1} \binom{i-1}{i_1} u^{(i-i_1)} u^{(i_1)} +$$

$$+ \sum_{i=3}^{n-1} \binom{n-1}{i} u^{(n-i)} \sum_{i_1=2}^{i-1} \binom{i-1}{i_1} u^{(i-i_1)} \sum_{i_2=1}^{i_1-1} \binom{i_1-1}{i_2} u^{(i_1-i_2)} \left(v_4 u^{(1)}\right)^{i_2-1} =$$

$$= v_1 u^{(n)} + v_2 \sum_{i=1}^{n-1} \binom{n-1}{i} u^{(n-i)} u^{(i)} + v_3 \sum_{i=2}^{n-1} \binom{n-1}{i} u^{(n-i)} \sum_{i_1=1}^{i-1} \binom{i-1}{i_1} u^{(i-i_1)} u^{(i_1)} +$$

$$+ v_4 \sum_{i=3}^{n-1} \binom{n-1}{i} u^{(n-i)} \sum_{i_1=2}^{i-1} \binom{i-1}{i_1} u^{(i-i_1)} \binom{i_1-1}{1} u^{(i_1-1)} u_1^{(1)} +$$

$$+ \sum_{i=4}^{n-1} \binom{n-1}{i} u^{(n-i)} \sum_{i_1=3}^{i-1} \binom{i-1}{i_1} u^{(i-i_1)} \sum_{i_2=2}^{i_1-1} \binom{i_1-1}{i_2} U^{(i_1-i_2)} \sum_{i_3=0}^{i_2-1} \binom{i_2-1}{i_3} u^{(i_2-i_3)} v_4^{(i_3)} =$$

$$= v_1 u^{(n)} + v_2 \sum_{i_0=1}^{n-1} \binom{n-1}{i} u^{(n-i)} u^{(i)} + v_3 \sum_{i=2}^{n-1} \binom{n-1}{i} u^{(n-i)} \sum_{i_1=1}^{i-1} \binom{i-1}{i_1} u^{(i-i_1)} u^{(i_1)} +$$

$$+ v_4 \sum_{i=3}^{n-1} \binom{n-1}{i} u^{(n-i)} \sum_{i_1=2}^{i-1} \binom{i-1}{i_1} u^{(i-i_1)} \binom{i_1-1}{1} u^{(i_1-1)} u^{(1)} +$$

$$+ v_4 \sum_{i=4}^{n-1} \binom{n-1}{i} u^{(n-i)} \sum_{i_1=3}^{i-1} \binom{i-1}{i_1} u^{(i-i_1)} \sum_{i_2=2}^{i_1 1} \binom{i_1-1}{i_2} u^{(i_1-i_2)} u^{(i_2)} +$$

$$+ \sum_{i=4}^{n-1} \binom{n-1}{i} u^{(n-i)} \sum_{i_1=3}^{i-1} \binom{i-1}{i_1} u^{(i-i_1)} \sum_{i_2=2}^{i_1-1} \binom{i_1-1}{i_2} U^{(i_1-i_2)} \sum_{i_3=0}^{i_2-1} \binom{i_2-1}{i_3} u^{(i_2-i_3)} \left(u^{(1)} v_5\right)^{(i_3-1)} =$$

$$= v_1 u^{(n)} + v_2 \sum_{i_0=1}^{n-1} \binom{n-1}{i_o} u^{(n-i_o)} u^{(i_o)} + v_3 \sum_{i_o=2}^{n-1} \binom{n-1}{i_o} u^{(n-i_o)} \sum_{i_1-1}^{i_o-1} \binom{i_o-1}{i_1} u^{(i_o-i_1)} u^{(i_1)} +$$

$$+ v_4 \sum_{i_o=3}^{n-1} \binom{n-1}{i_o} u^{(n-i_o)} \sum_{i_1=2}^{i_o-1} \binom{i_o-1}{i_1} u^{(i_o-i_1)} \sum_{i_2=1}^{i_1-1} \binom{i_1-1}{i_2} u^{(i_1-i_2)} u_1^{(i_2)} +$$

$$+ v_5 \sum_{i_o=4}^{n-1} \binom{n-1}{i_o} u^{(n-i_o)} \sum_{i_1=3}^{i_o-1} \binom{i_o-1}{i_1} u^{(i_o-i_1)} \sum_{i_2=2}^{i_1-1} \binom{i_1-1}{i_2} u^{(i_1-i_2)} \binom{i_2-1}{1} u_1^{(i_2-1)} u^{(1)} +$$

$$+ \sum_{i=5}^{n-1} \binom{n-1}{i_o} u^{(n-i_o)} \sum_{i_1=3}^{i_o-1} \binom{i_o-1}{i_1} u^{(i_o-i_1)} \sum_{i_2=3}^{i_1-1} \binom{i_1-1}{i_2} u^{(i_1-i_2)} \sum_{i_3=2}^{i_2-1} \binom{i_2-1}{i_3} u^{(i_2-i_3)} \sum_{i_4=0}^{i_3-1} \binom{i_3-1}{i_4} u^{(i_3-i_4)} v_5^{(i_4)}$$

Continuing this derivation we have

$$M^{(n)} = v_1 u^{(n)} + v_2 \sum_{i_o=1}^{n-1} \binom{n-1}{i_o} u^{(n-i_o)} u^{(i_o)} + v_3 \sum_{i_o=2}^{n-1} \binom{n-1}{i_o} u^{(n-i_o)} \sum_{i_1=1}^{i_o-1} \binom{i_o-1}{i_1} u^{(i_o-i_1)} u^{(i_1)} +$$

$$+ v_4 \sum_{i_o=3}^{n-1} \binom{n-1}{i_o} u^{(n-i_o)} \sum_{i_1=2}^{i_o-1} \binom{i_o-1}{i_1} u^{(i_o-i_1)} \sum_{i_2=1}^{i_1-1} \binom{i_1-1}{i_2} u^{(i_1-i_2)} u^{(i_2)} + \ldots$$

$$\ldots + v_k \sum_{i_o=k-1}^{n-1} \binom{n-1}{i_o} u^{(n-i_o)} \sum_{i_1=k-2}^{i_o-1} \binom{i_o-1}{i_1} u^{(i_o-i_1)} \ldots \sum_{i_h=k-h-1}^{i_{h-1}-1} \binom{i_{h-1}-1}{i_h} u^{(i_{h-1}-i_h)}$$

$$\sum_{i_{h+1}=k-h-2}^{i_h-1} \binom{i_h-1}{i_{h+1}} u^{(i_h-i_{h+1})} \ldots \sum_{i_{k-3}=2}^{i_{k-4}-1} \binom{i_{k-4}-1}{i_{k-3}} u^{(i_{k-4}-i_{k-3})} \sum_{i_{k-2}=1}^{i_{k-3}-1} \binom{i_{k-3}-1}{i_{k-2}} u^{(i_{k-3}-i_{k-2})} u^{(i_{k-2})} + \ldots$$

$$+ \ldots v_{n-2} \sum_{i_o=n-3}^{n-1} \binom{n-1}{i_o} u^{(n-i_o)} \sum_{i_1=n-4}^{i_o-1} \binom{i_o-1}{i_1} u^{(i_o-i_1)} \sum_{i_2=n-5}^{i_1-1} \binom{i_1-1}{i_2} u^{(i_1-i_2)} \ldots$$

$$\ldots \sum_{i_h=n-h-3}^{i_{h-1}-1} \binom{i_{h-1}-1}{i_h} u^{(i_{h-1}-i_h)} \ldots \sum_{i_{n-4}=1}^{i_{n-5}-1} \binom{i_{n-5}-1}{i_{n-4}} u^{(i_{n-5}-i_{n-4})} u^{(i_{n-4})} +$$

$$+ v_{n-1} \sum_{i_o=n-2}^{n-1} \binom{n-1}{i_o} u^{(n-i_o)} \sum_{i_1=n-3}^{i_o-1} \binom{i_o-1}{i_1} u^{(i_o-i_1)} \ldots \sum_{i_h=n-2-h}^{i_{h-1}-1} \binom{i_{h-1}-1}{i_h} u^{(i_{h-1}-i_h)} \ldots$$

$$\ldots \sum_{i_{n-3}=1}^{i_{n-4}-1} \binom{i_{n-4}-1}{i_{n-3}} u^{(i_{n-4}-i_{n-3})} u^{(i_{n-3})} + v_n \left(u^{(1)}\right)^n$$

This result can be written in a more synthetic way by means of the expression (1.8).

Appendix 3.2.

Let I_n be the unit matrix of the degree n. The matrices D_m and $D_{s_i}^{(h)}$,are diagonal ones of degree n, i = 1, 2, ..., r. Their diagonal elements can be equal zero or one. Moreover:

$$D_m + \sum_{i=1}^{r} D_{s_i}^{(h)} = I_n, \quad h = 1, \ldots, P(s_1, \ldots, s_r) \tag{3.3}$$

where $0 \le m \le n$ and

$$\underset{i=1,\dots,r}{\Lambda} \quad 0 \le s_i \le n-m \quad i \quad \sum_{j=1}^{r} s_j = (n-m) = s \tag{3.4}$$

$$P(s_1, s_2, \dots, s_r) = s! / \prod_{i=1} s_i \; ! \tag{3.5}$$

The matrix $D_{s_i}^{(h)}$, i = 1, 2, ..., r, has s_i – diagonal elements equal to one and $(n-s_i)$ diagonal elements equal to zero. Similarly, the matrix D_m has m and (n-m) its diagonal elements equal to one and zero, respectively. Under the fixed matrix D_m and fixed numbers s_1, s_2, ..., s_r the number of different the sequence of the matrices $D_{s_1}^{(h)}$, ..., $D_{s_r}^{(h)}$ is equal to number of permutation of a sequence consisted of r elements replicated s_1, s_2, ..., s_r times and $\sum_{j=1}^{r} s_j = s = n - m$ and $s_j \ge 0$, j = 1, ..., r. The number of these permutation is denoted by $P(s_1,s_2, \dots, s_r)$ and determined by the expression (3.5)

Let each matrix of the sequence A, A_1, ..., A_r is of degree n and:

$$W_h\left(s_1, s_2, \dots, s_r \mid m\right) = D_m A + \sum_{i=1}^{r} D_{s_i}^{(h)} A_i \tag{3.6}$$

$$K_h\left(s_1, s_2, \dots, s_r \mid m\right) = A D_m + \sum_{i=1}^{r} A_i D_{s_i}^{(h)} \tag{3.7}$$

where $h=1,2,\dots,P(s_1,s_2,\dots,s_r)$. The matrix $W_h\left(s_1, s_2, \dots, s_r \mid m\right)$ consists of m, s_1, s_2, ..., s_r rows of the matrices A, A_1, A_2, ..., A_r respectively.

The following well-known in linear algebra theorems will be considered:

Theorem 3.1. If for the i-th row of a matrix A of the degree n the following equation is fulfilled

$$[a_{i1}\ a_{i2}\ \dots\ a_{in}] = x_1\, [b_{i1}\ b_{i2}\ \dots\ b_{in}] + x_2\, [c_{i1}\ c_{i2}\ \dots\ c_{in}]$$

then

$$\det A = x_1 \det E_1 + x_2 \det E_2$$

where the matrix E_1 is obtained from the matrix A through substituting its i-th row for the vector $[b_{i1}\ b_{i2}\ \dots\ b_{in}]$. Similarly, when we substitute the i-th row of the matrix A for the row $[c_{i1}\ c_{i2}\ \dots\ c_{in}]$ we have the matrix E_2.

Let us consider the following theorem:

Theorem 3.2. (Wywiał, 1990) Let each of the matrices A, A_1, A_2, ..., A_r is of degree n and x_1, x_2, ..., x_r are real numbers. If it is fulfilled the equation:

$$A = \sum_{i=1}^{r} A_i x_i \tag{3.8}$$

then for $m = n - s \geq 0$ we have

$$\det A = \sum_{s_1=0}^{u_1} \sum_{s_2=0}^{u_2} \dots \sum_{s_i=0}^{u_i} \dots \sum_{s_{r-1}}^{u_{r-1}} L(s_1, s_2, ..., s_r \mid m) \prod_{i=1}^{r} x_i^{s_i} \tag{3.9}$$

where:

$$L(s_1, s_2, ..., s_r \mid m) = \sum_{h=1}^{P(s_1,...,s_r)} \det W_h(s_1, s_2, ..., s_r \mid m) \tag{3.10}$$

or:

$$L(s_1, s_2, ..., s_r \mid m) = \sum_{h=1}^{P(s_1,...,s_r)} \det K_h(s_1, s_2, ..., s_r \mid m) \tag{3.11}$$

$$u_i = n - m - \sum_{j=1}^{i-1} s_j; \quad i = 1,2,...,r-1 \tag{3.12}$$

The matrices W_h and K_h and s, s_1, s_2, ..., s_r, P(...) are determined by the expressions (3.6), (3.7) and (3.4), (3.5).

Prove: For simplification of the derivation let us assume that for i = 1, 2, ..., r, $x_1 = 1$. Firstly, the induction prove will be started for r = 2, and next for r > 2.

The theorem 3.1 leads to the conclusion that if r = 2 and s = 1, then:

$$\det A = W_1(1,0 \mid n\text{-}1) + W_2(0,1 \mid n\text{-}1)$$

where only i-th row in the matrices W_1 (...), W_2 (...) and A are different. These rows are equal to: $\left[a_{i1}^{(1)} + a_{i1}^{(2)} a_{i2}^{(1)} + a_{i2}^{(2)} ... a_{in}^{(1)} + a_{in}^{(2)} \right]$, $\left[a_{i1}^{1} a_{i2}^{(1)} ... a_{in}^{(1)} \right]$ and $\left[a_{i1}^{(2)} a_{i2}^{(2)} ... a_{in}^{(2)} \right]$ in the matrices A, W_1 and W_2, respectively.

We are going to prove that if the equation (3.9) is valid for dla r = 2 and $1 \le s < n$, then it is valid for s + 1. Hence, the right hand of the equation (3.9) is the induction assumption. On the basis of the theorem 3.1 the determinant $W_h(s_1,s-s_1|m)$ is decomposed into the sum of two following matrices $W_h(s_1+1, s-s_1|m-1)$ and $W_h(s_1,s-s_1+1|m-1)$. Let us assume that the i-th row of the matrix $W_h(s_1,s-s_1|m)$ is equal to the i-th following row of the matrix A:

$$\left[a_{i1}, a_{i2} ... a_{in}\right] = \left\lfloor a_{i1}^{(1)} a_{i2}^{(1)} ... a_{n}^{(1)} \right\rfloor + \left\lfloor a_{i1}^{(2)} a_{i2}^{(2)} ... a_{in}^{(2)} \right\rfloor$$

Hence, the matrix $W_h(s_1+1, s-s_1|m-1)$ has the i-th row equal to the i-th row $\left\lfloor a_{i}^{(1)} a_{i2}^{(1)} ... a_{in}^{(1)} \right\rfloor$ of the matrix A_1. The i-th row $\left\lfloor a_{i1}^{(2)} a_{i2}^{(2)} ... a_{in}^{(2)} \right\rfloor$ of the matrix A_2 is equal to the i-th row of the matrix $W_h(s_1,s-s_1+1|m-1)$. Hence, in these two matrices there is (m-1) rows from the matrix A and (s+1) rows from the matrices A_1 and A_2. Thus, let us rewrite the equation (3.9) in the following way:

$$\det A = \sum_{s_1=0}^{s} \sum_{h=0}^{\binom{s}{s_1}} \left\{\det W_h\left(s_1+1, s-s_1 \mid m-1\right) + \det W_h\left(s_1, s-s_1+1 \mid m-1\right)\right\}$$

For a moment in the following derivation we neglect the index m-1.

$$\det A = \sum_{s_1=0}^{s} \sum_{h=0}^{\binom{s}{s_1}} \det W_h\left(s_1+1, s-s_1\right) + \sum_{s_1=0}^{s} \sum_{h=0}^{\binom{s}{s_1}} \det W_h\left(s_1, s-s_1+1\right)$$

$$\det A=\Bigg[\sum_{h=0}^{\binom{s}{0}}\det W_h(1,s)+\sum_{h=0}^{\binom{s}{1}}\det W_h(2,s-1)+\ldots+\sum_{h=0}^{\binom{s}{k}}\det W_h(k+1,s-k)+$$

$$+\sum_{h=0}^{\binom{s}{k+1}}\det W_h(k+2,s-k+1)+\ldots+\sum_{h=0}^{\binom{s}{s-2}}\det W_h(s-1,2)+\sum_{h=0}^{\binom{s}{s-1}}\det W_h(s,1)+$$

$$+\sum_{h=0}^{\binom{s}{s}}\det W_h(s+1,0)\Bigg]+\Bigg[\sum_{h=0}^{\binom{s}{0}}\det W_h(0,s+1)+\sum_{h=0}^{\binom{s}{1}}\det W_h(1,2)+$$

$$+\sum_{h=0}^{\binom{s}{2}}\det W_h(2,s-1)+\ldots+\sum_{h=0}^{\binom{2}{k+1}}\det W_h(k+1,s-k)+\sum_{h=0}^{\binom{s}{k+2}}\det W_h(k+2,s-k-1)+$$

$$+\ldots\sum_{h=0}^{\binom{s}{2-1}}\det W_h(s-1,2)+\sum_{h=0}^{\binom{s}{s}}\det W_h(s,1)\Bigg]$$

This and the equation $\binom{n}{v}+\binom{n}{v-1}=\binom{n+1}{v}$ lead to the following expression:

$$\det A=\sum_{h=0}^{\binom{s+1}{0}}\det W_h(0,s+1)+\sum_{h=0}^{\binom{s+1}{k+2}}\det W_h(1,s)+\ldots+\sum_{h=0}^{\binom{s+1}{k+1}}\det W_h(k+1,s-k)+$$

$$+\sum_{h=0}^{\binom{s+1}{k+2}}\det W_h(k+2,s-k-1)+\ldots+\sum_{h=0}^{\binom{s+1}{s-1}}\det W_h(s-1,2)+$$

$$+\sum_{h=0}^{\binom{s+1}{s}}\det W_h(s,1)+\sum_{h=0}^{\binom{s+1}{s+1}}\det W_h(s+1,0)$$

This equation with the index (m-1) can be rewritten as follows:

$$\det A = \sum_{s_1=0}^{s+1} \sum_{h=0}^{\binom{s+1}{s_1}} \det W_h\left(s_1, s-s_1+1 \mid m-1\right) \tag{3.13}$$

Hence, for r = 2 and $x_i = 1$, i = 1, 2, the equation (3.9) is valid for s and s+1. This result is useful for proving the expression (3.9) for r > 2.

Let us substitute the matrix A_r' for the matrix A_r in the equation (3.8). Moreover, let us assume that:

$$A_r' = A_r + A_{r+1} \tag{3.14}$$

Let the right hand of the equation (3.9) be the induction assumption. It can be rewritten in the following way:

$$\det A = \det\left(\sum_{i=1}^{r-1} A_i + A_r'\right) = \sum_{s_1=0}^{s} \sum_{s_2=0}^{u_2} \ldots \sum_{s_{r-1}=0}^{u_{r-1}} \sum_{h=0}^{P(s_1 \ldots s_r)} \det W_h\left(s_1 . s_2, \ldots . s_{r-1}, s - \sum_{i=1}^{r-1} s_i \mid m\right) \tag{3.15}$$

Each matrix $W_h\left(s_1, s_2, \ldots, s_{r-1}, s - \sum_{i=1}^{r-1} s_1 \mid m\right)$ is decomposed according to the expression (3.13) provided only the matrix A_r' is decomposed. The matrices A_i, i =1, 2, ..., r-1 are not changed in this decomposition. This leads to the following expression:

$$\det W_h\left(s_1, s_2, \ldots, s_{r-1}, s - \sum_{i=1}^{r-1} s_i \mid m\right) = \sum_{s_r=0}^{u_r} \sum_{l=0}^{\binom{u_r}{s_r}} \det W_h\left(s_1, s_2, \ldots, s_{r-1}, s_r, s - \sum_{i=1}^{r} s_i \mid m\right) \tag{3.16}$$

where u_r is defined by the equation (3.12) for i = r. The right hand of this equation and the expression (3.15) lead to the following ones:

$$\det A = \sum_{s_1=0}^{s} \sum_{s_2=0}^{u_2} \ldots \sum_{s_i=0}^{u_i} \ldots \sum_{s_{r-1}=0}^{u_{r-1}} \sum_{h=0}^{P(s_1, \ldots, s_r)} \sum_{s_r=0}^{u_r} \sum_{l=0}^{\binom{u_r}{s_r}} \det W_{hl}\left(s_1, s_2, \ldots, s_{r-1}, s_r, s - \sum_{i=1}^{r} s_i \mid m\right)$$

$$\det A = \sum_{s_1=0}^{s} \sum_{s_2=0}^{u_2} \ldots \sum_{s_i=0}^{u_i} \ldots \sum_{s_{r-1}=0}^{u_{r-1}} \sum_{s_r=0}^{u_r} \sum_{h=0}^{P(s_1, \ldots, s_r)} \sum_{l=0}^{\binom{u_r}{s_r}} \det W_{hl}\left(s_1, s_2, \ldots, s_r, s - \sum_{i=1}^{r} s_i \mid m\right) \tag{3.17}$$

The number of the elements of the sum in the square brackets is equal to $\binom{u_r}{s_r}P(s_1,...,s_r)$. On the basis of the expressions (3.5) and (3.12) we have:

$$P(s_1,s_2,...,s_{r+1}) = s!/\prod_{i=1}^{r+1} s_i! = \frac{s!}{s_1!s_2!...s_r!\left(s-\sum_{i=1}^{r} s_i\right)!} =$$

$$= \frac{s!}{s_1!s_2!...s_{r-1}!\left(s-\sum_{i=1}^{r-1} s_i\right)!} \frac{\left(s-\sum_{i=1}^{r-1} s_i\right)!}{s_r!\left(s-\sum_{i=1}^{r} s_i\right)!} =$$

$$= P(s_1,s_2,...,s_{r-1})\frac{u_r!}{s_r!(u_r-s_r)!} = \binom{u_r}{s_r}P(s_1,...,s_r)$$

This allows us to rewrite the right hand of the equation (3.17) in the following way:

$$\det A = \sum_{s_1=0}^{s}\sum_{s_2=0}^{u_2}...\sum_{s_i=0}^{u_i}...\sum_{s_{r-1}=0}^{u_{r-1}}\sum_{s_r=0}^{u_r}\sum_{h=0}^{P(s_1,...,s_{r+1})}\det W\left(s_1,...,s_r,s-\sum_{i=1}^{r} s_i \mid m\right)$$

Hence, the expression (3.9) is valid for $x_i = 1$, i = 1, 2, ..., r because it is valid for r+1 under the assumption that it is valid for r.

This proof can be easy generalised in the case when numbers $x_i \neq 1$, i = 1, 2, ..., r . In the similar way we can prove the expression (3.9), when $L(s_1, s_2, ..., s_r|m)$ are determined by the equation (3.11).

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