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Grażyna Trzpiot

MULTICRITERION NONCLASSICAL MODELING BASED ON MULTIVALUED STOCHASTIC DOMINANCE AND PROBABILISTIC DOMINANCE IN CAPITAL MARKET

1. Introduction

According to the expected utility paradigm, the risk of the investment must be related to the assumed preference of the investor and cannot be objectively defined. A natural quantitative definition of the risk is the amount of money one is willing, on average, to pay someone else to assume the risk. In the empirical study we have problem with ranking alternatives in the area of the financial issues. The classical method for ranking the alternatives is based on comparing means and variances of two alternatives (mean-variance criterion) [6]. Rothschild and Stiglitz suggest new look on the risk for two distributions characterised by the same expected return [8]. They used the stochastic dominance rules for describing more risky asset. In the empirical study when we do not observe the stochastic dominance, we can use additionally a probabilistic dominance.

The multicriterion formulation of a decision situation can be defined as a model of three components: the set of attributes, the set of actions and the set of evaluations. Each pair (attribute, action) is described by a vector of evaluation, which may be of different nature.

In our work this set of vectors has the multivalued random components. In multicriterion analysis under uncertainty we apply the stochastic dominance and probabilistic dominance procedures in the decision making process on the capital market, in the case of multivalued alternatives. We have made an empirical analysis by this multicriterion multivalued model: evidence from the Warsaw Stock Exchange.

2. Multivalued Stochastic Dominance

Let F and G are the cumulative distribution functions of two distinct uncertain alternatives X and Y. X dominates Y by first, second and third stochastic dominance (FSD, SSD, TSD) if and only if

$$H_1(x) = F(x) - G(x) \le 0 \text{ for all } x \in [a,b] \ \ (X \text{ FSD } Y) \tag{2.1}$$

$$H_2(x) = \int_a^x H_1(y)dy \le 0 \text{ for all } x \in [a,b] \ \ (X \text{ SSD } Y) \tag{2.2}$$

$$H_3(x) = \int_a^x H_2(y)dy \le 0 \text{ for all } x \in [a,b]$$
$$\text{and } E(F(x)) \ge E(G(x)) \text{ (X TSD Y)} \tag{2.3}$$

The relationship between the three stochastic dominance rules can be summarised by the following diagram: FSD ⇒ SSD ⇒ TSD, which means that the dominance by FSD implies the dominance by SSD and dominance by SSD in turn implies the dominance by TSD. For proof of FSD and SSD see Hadar and Russell [1], Hanoch and Levy [2] and Rothschild and Stiglitz [8]. The criterion for TSD was suggested by Whitmore [13].

When we have ambiguities in the probabilities and outcomes, we have no single – valued distribution, such a situation can be represented by a set of probability distributions. Each family has two border (lower and upper) probability distributions functions the scalar outcome space X.

Definition 1. Lower probability distributions for all values $x_i \in X$, we call probability count as

$$p_*(x_i) = \sum_{j:x_i=\min\{y:y\in A_j\}} p(A_j), \tag{2.4}$$

According to this definition we have: $\sum_i p_*(x_i) = 1$.

Definition 2. Upper probability distributions for all values $x_i \in X$, we call probability count as

$$p^*(x_i) = \sum_{j:x_i=\max\{y:y\in A_j\}} p(A_j), \tag{2.5}$$

Now, we also have: $\sum_i p^*(x_i) = 1$.

In case of the point values of the random variables both distributions (lower and upper probability distributions) are exactly the same: $p_*(x_i) = p^*(x_i) = p(x_i)$ and we have the probability distributions in classical sense

Example 2.1.

We determine the lower and upper probability distributions for random variable **X**, whose outcomes are multivalued, in some intervals:

A_j	[2, 4]	[3, 4]	[4, 5]	[5, 6]
$p(A_j)$	0,2	0,5	0,2	0,1

According to the definitions 1 and 2, we have lower and upper probability distributions for a random variable **X**:

x_j	2	3	4	5	6
$p_*(x_j)$	0,2	0,5	0,2	0,1	-
$p^*(x_j)$	-	-	0,7	0,2	0,1

Using these lower and upper probability distributions for random variables, which outcomes are multivalued, we can make a ranking such a multivalued alternative.

♦

Let two distinct uncertain multivalued alternatives **X** and **Y** have the lower cumulative distributions respectively $F_*(x)$ and $G_*(x)$, upper cumulative distributions respectively $F^*(x)$ and $G^*(x)$, for $x \in [a, b]$, then we have multivalued first, second and third stochastic dominance if and only if

$$H_1(x) = F_*(x) - G^*(x) \le 0 \text{, for all } x \in [a,b] \quad , \quad (\mathbf{X} \text{ FSD } \mathbf{Y}) \tag{2.6}$$

$$H_2(x) = \int_a^x H_1(y)dy \le 0 \text{, for all } x \in [a,b] \quad , \quad (\mathbf{X} \text{ SSD } \mathbf{Y}) \tag{2.7}$$

$$H_3(x) = \int_a^x H_2(y)dy \le 0 \text{, for all } x \in [a,b] \quad , \quad (\mathbf{X} \text{ TSD } \mathbf{Y}) \tag{2.8}$$

$$\text{oraz } E(F_*(x)) \ge E(G^*(x))$$

For proof see Langewisch and Choobineh [3].

Example 2.2.

Let us take a random variable **C** and **D** which outcomes are multivalued, in some intervals as follows:

A_j	[0,1]	[1, 2]	[2, 3]	[3, 4]
p(**C**)		0,2	0,4	0,4
p(**D**)	0,3	0,15	0,55	-

We determine the lower and upper probability distributions for random variables **C** and **D** as follows:

x_j	0	1	2	3	4
C: $p_*(x_j)$	-	0,2	0,4	0,4	-
C: $p^*(x_j)$	-	-	0,2	0,4	0,4
D: $p_*(x_j)$	0,3	0,15	0,55	-	-
D: $p^*(x_j)$	-	0,3	0,15	0,55	-

Now we can receive the values of the lower and upper cumulative distributions.

x_j	$(-\infty,0]$	(0,1]	(1, 2]	(2, 3]	(3,4]	$(4, \infty)$
$C_*(x_j)$	0	0	0,2	0,6	1	1
$C^*(x_j)$	0	0	0	0,2	0,6	1
$D_*(x_j)$	0	0,3	0,45	1	1	1
$D^*(x_j)$	0	0	0,3	0,45	1	1

It is easy to check, that the definition 2.6 and the definition 2.7.are not satisfying. In this example we can establish a third degree multivalued stochastic dominance: **C** TSD **D**.

♦

3. Multivalued Probability Dominance

The dominance concepts discussed previously are based on a theoretical foundation of utility function. However, if methods usually elaborate for the problem of the risky choice are too far away from our habitual thinking, they may likely be rejected in practice. For this purpose, Wrather and Yu [11] introduced a probability dominance concept, which is based on a habitual way of thinking in making decision. We started from the study of rational behaviour of investors. We have two uncertain alternatives, represented by multivalued random variables **X** and **Y**, and we observe that

1. the random variables **X** and **Y** do not stochastically dominate each other
2. the random variables **X** and **Y** do not dominate each other in the sense of mean-variance,
3. the investor believes that random variables **X** is better than **Y**.

We can add statement notice like investor believes that the random variables **X** dominate **Y** in the probability dominance sense.

Definition. 3. Given two multivalued random variables **X** and **Y**, we say that **X** dominates **Y** with probability $\beta \geq 0,5$ (X PD Y), iff $P(X_* > Y^*) \geq \beta$.

$P(X_* > Y^*)$ is probability, that X_* outperforms Y^* and β is alike a measure. If $\beta > 0,5$, then $P(Y^* > X_*) \leq 1 - \beta < 0,5$. So in this case **X** β **Y** means, that **X** is as good as **Y** in 50%.

Example. 3.1

Let **X** and **Y** be two random multivalued variables with such lower and upper probability distributions: $P(X_* = 0,1) = 1/4$, $P(X_* = 0,5) = 1/2$, $P(X_* = 1) = 1/4$, $P(Y^* = 0,2) = 1/2$ $P(Y^* = 0,3) = 1/2$. We can see that $X_* < Y^*$ in 25% of value and $X_* > Y^*$ in 75% of value. We can write that $P(X_* > Y^*) = P(X_* = 0,5) + P(X_* = 1) =$ $= 1/2 + 1/4 = 3/4$.

That means X β Y for any $\beta \in [0,5; 0,75]$. In this example we observe that:

1. the random variables X_* and Y^* do not stochastically dominate each other in either FSD and SSD
2. the random variables X_* and Y^* do not dominate each other in the sense of mean-variance,
3. the random variable X_* dominate Y^* with probability $\beta \in [0,5; 0,75]$.

This two different concepts of dominance: stochastic dominance and probability dominance can be complementary to each other. Some clear dominance between the random outcomes can be found by using the stochastic dominance test or by using probability dominance test.

4. Multiattribute Multivalued Problem

Multiattribute problem can be represented as a model *A, X, E* (Alternatives, Attributes, Evaluations). We propose a multivalued case of a method presented by Martel and Zaraś [7]. We have:

1. a finite set of a*lternatives* (actions) $A = \{a_1, a_2,..., a_m\}$;
2. a finite set *of attributes* $X = \{X_1, X_2,..., X_n\}$, which are independent and we can apply the additive utility function;
3. a set of evaluations $E = \{X_{ij}\}_{mxn}$, where X_{ij} is a random multivalued variable with an lower and upper density functions $[{}_l f_{ij}(x), {}_u f_{ij}(x)]$. The interval of variation associated with the attribute X_i is denoted by $[x_{io}, x_i^o]$.

Value of each alternatives according to each attributes can be represented as a random variable with a density function $f_{ij}(x)$.

Table 1

Model (A, X, E) for multiattribute problem

Attributes Alternatives	X_1	X_2	X_j	X_n
a_1	$[{}_l f_1(x_{11}), {}_u f_1(x_{11})]$	$[{}_l f_1(x_{12}), {}_u f_1(x_{12})]$	$[{}_l f_1(x_{1j}), {}_u f_1(x_{1j})]$	$[{}_l f_1(x_{1n}), {}_u f_1(x_{1n})]$
a_2	$[{}_l f_2(x_{21}), {}_u f_2(x_{21})]$	$[{}_l f_2(x_{22}), {}_u f_2(x_{22})]$	$[{}_l f_2(x_{2j}), {}_u f_2(x_{2j})]$	$[{}_l f_2(x_{2n}), {}_u f_2(x_{2n})]$
....	....	....	....	....
a_m	$[{}_l f_m(x_{m1}),{}_u f_m(x_{m1})]$	$[{}_l f_m(x_{m2}),{}_u f_m(x_{m2})]$	$[{}_l f_m(x_{mj}),{}_u f_m(x_{mj})]$	$[{}_l f_m(x_{mn}),{}_u f_m(x_{mn})]$

We compare each two cumulative distributions for the determine multivalued stochastic dominance. After that for perception the difference between X_* FSD Y^* and $P(X_*>Y^*) = 0{,}95$ and X_* FSD Y^* and $P(X_*>Y^*) = 0{,}05$ we would like to suggest a precriterion.

We will model the local preference by two binary relations based on multivalued probability dominance: ***P*** (strict preference) and ***Q*** (weak preference) as follows:

1. X_* ***P*** Y^* if $\exists\, x\alpha \in X_*$ such that $P(Y^* < x\alpha) > \beta / (1- \alpha)$ and $\neg\, Y^*$ SD X_*, where $\beta \in [0{,}5; 1]$, $\alpha \in [0; 1)$ and $x\alpha = \sup \{xk: P(X_* < xk) \le \alpha\}$
2. X_* ***Q*** Y^* if $\forall\, x\alpha \in X_*$ such that $P(Y^* < x\alpha) \le \beta / (1- \alpha)$ and X_* SD Y^*
3. X_* ***N*** Y^* for the others (means incomparability and indifference relations).

Finally we would like to build a global preference relationship between each pair in a multiattribute problem involving the risk. The suggested procedure for aggregating local preferences is as follows:

$a_i \prec a_j$ if $\neg\, a_i\, \boldsymbol{P}\, a_j$ for all k if $wP+ + wQ+ \geq wQ-$

$a_i \sim a_j$ for the others;

where two binary relations $\prec$ and $\sim$ are defined respectively as large preference and no preference and

$wP+$ is the sum of weigths for all k where $a_i\, \boldsymbol{P}\, a_j$

$wQ+$ is the sum of weigths for all k where $a_i\, \boldsymbol{Q}\, a_j$

$wQ-$ is the sum of weigths for all k where $a_i\, \boldsymbol{Q}\, a_j$.

5. An empirical application of a Multiattribute Multivalued Model

We take for our analysis the empirical results from the Warsaw Stock Exchange. This illustration is based on four best assets – a set of four *alternatives*. As a finite set of two *attributes* we took the daily return, which is a random multivalued variable (the random value in the interval between minimum and maximum return) and the volume. We were took the observations from 2.04.2002 to 28.06.2002. After building a random multivalued variable we observed the multivalued stochastic dominance (table 2).

Table 2

Observed multivalued stochastic dominance

Attribute 1	ELZ	FTE	MSW	ZWC
ELZ	X			
FTE	FSD	X	=	=
MSW	FSD	=	X	=
ZWC	FSD	=	=	X

Attribute 2	ELZ	FTE	MSW	ZWC
ELZ	X			
FTE	FSD	X		
MSW	FSD	FSD	X	
ZWC	FSD	FSD	FSD	X

Because, we observed the same class of multivalued stochastic dominance (this is a part of FSD effective set), so we need to establish the multivalued probability dominance and next the local preference (table 3 and 4), most of them are weak preferences.

Table 3

Observed multivalued probability dominance

Attribute 1	ELZ	FTE	MSW	ZWC
ELZ	X	0,016	0,016	0,016
FTE		X	0	0
MSW		0	X	0
ZWC		0	0	X

Attribute 2	ELZ	FTE	MSW	ZWC
ELZ	X	0,048	0,049	0,049
FTE		X	0,033	0,033
MSW			X	0,016
ZWC				X

Table 4

Observed multivalued precriterion

Attribute 1	ELZ	FTE	MSW	ZWC
ELZ	X			
FTE	***Q***	X		
MSW	***Q***		X	
ZWC	***Q***			X

Attribute 2	ELZ	FTE	MSW	ZWC
ELZ	X			
FTE	***Q***	X		
MSW	***Q***	***Q***	X	
ZWC	***Q***	***Q***	***Q***	X

Finally we would like to build a global preference relationship between each pair in a multiattribute model (table 5). Next we can draw a graph based on ELECTRE I method, which illustrate our global preference relationship. Using this graph (figure 1 – reading from left to right) we can easily establish a global preference in our set of the alternatives. The best is 4-ZWC, next 3-MSW, 2-FTE and the worst is 1-ELZ.

Table 5

Observed global preference relationship

Global preference	ELZ	FTE	MSW	ZWC
ELZ	X	~	~	~
FTE	>	X	~	~
MSW	>	>	X	~
ZWC	>	>	>	X

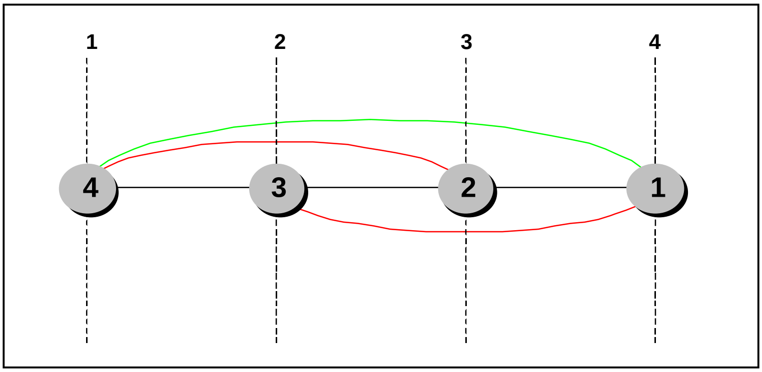

Fig. 1. Observed global preference relationship

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