

The Karol Adamiecki University of Economics  
in Katowice

Volume 3

2007

*Journal of*

---

**Economics &  
Management**

*Włodzimierz Szkutnik*

---

**MODELLING OF EXTREME  
VALUES RISK IN  
ATYPICAL CONDITIONS  
OF A LOSS PROCESS REALIZATION**

In the first approach to implementation of appropriate formal procedures reality of an insurance market to be discussed referred to a process of losses and to a measure of risk expressed probabilistically [1]. Practical considerations and a possibility to utilise formal discussions on the process of losses while analysing processes of reserves and claims resulted in changing the way this very aspect of insurance risk is perceived. Therefore the paper firstly presents risk of reaching a critical value by reserves that are formed to provide compensations. Secondly it discusses risk associated with realising high reserves and claims modelled by  $\alpha$  – stable distributions.

## **1. Model of risk associated with reserves in the process of unpaid compensations and benefits**

Issues discussed in the paper refer to modelling of risk that emerges when reserves and claims reach certain values thought to be critical in relation to some empirical arrangements or arrangements set by decision makers.

In this situation it is not important how large a group of unpaid compensations in a given portfolio within some period of time is. Let us consider the following formal scheme of reserves created to cover unpaid compensations and benefits that relate to potential high claims. A value of reserves aimed at paying out compensations plays a major role here. It is also essential to know when the reserves in question will deplete. The model that is being discussed may be applied in order to find proportions while determining risk reinsurance and to assess risk involved while concluding such contracts.

Let  $X_i$  mean reserves value in an  $i$  period. Then it is possible to determine risk of exceeding or not exceeding certain given required value by reserve value in each period of the analysis e.g. in the inter-premium period. This risk is described by a random variable that which expresses a minimum of values reached by reserves that are random variables in subsequent  $K$  periods. It is determined by the following minimisation:

$$X_{Kmin} = \min \{X_1, X_2, \dots, X_K\}$$

that sets a minimal value of the reserve in  $K$  periods per one compensation or per policies included in the policy portfolio. In order to fix model conditions let us assume here that a critical situation for financial security of an insurance com-

pany is when for each policy of the portfolio or in each analysed period the value of reserves reaches a critical level  $x$  for the process of compensation payout.

Variables  $X_i$  may be treated as independent. It is going to be considered on further on. These variables are obviously random variables determined by a random character of loss being brought about in a particular policy of the homogeneous portfolio of policies and by existence of incidental changes observed in the phenomenon of (a process of) emerging loss in a given risk. We will also assume that  $X_i$  has the same distribution determined by  $\Psi$  distribution function. This way we have reached a model situation that may be applied in a risk associated with emergence of a given number of loss in given time. We have also set a simultaneous adjustment of loss reserves in all portfolios of an insurance company or so called "post-loss" reinsurance. However, time is now measured in a number of periods or in other discrete units. Some attention has to be paid to the differences between the issues that are being discussed now and the ones that have already been signposted. The major difference is not in a dissimilarity of the discussed concepts (they may be easily standardised) but in the value of  $K$  number of random variables in minimisation. This number may now be very large. This fact will allow for using some expressions obtained in border realities of a model with  $K \rightarrow \infty$ .

Let us introduce the following symbols and assumptions.

$\Psi(x)$  – cumulative distribution function of  $X_i$  reserves values in the  $i$  period. Then on the basis of a definition of a cumulative distribution function and an assumption of independence we obtain:

$$P(X_1 < x) = P(X_2 < x) = \dots = P(X_K < x) = \Psi(x)$$

and therefore using a definition of  $X_{K \min}$  variable we obtain the following formula:

$$P(X_{K \min} \geq x) = P(X_1 \geq x, \dots, X_K \geq x) = [1 - \Psi(x)]^K \quad (1)$$

In connection with a previously adopted assumption the relation (1) determines a risk level of shaking financial equilibrium or emergence of conditions that threat portfolio financial security.

Risk that is going to be discussed further on concerns reaching unwanted value by the level of reserves. In each period the reserve is higher that a given number  $X_0 \geq 0$ . Then the following equality (ralation) must occur:

$$P(X_i < X_0) = \Psi(X_0) = 0$$

What is more, because of formal reasons we might also assume that for any  $h > 0$  the following inequality occurs:

$$P(X_i < X_o + h) = \Psi(X_o + h) > 0$$

However for low values  $h$  the following equality is true:

$$P(X_i < X_o + h) = \Psi(X_o + h) = (c + \varepsilon_h) \cdot h^\alpha \tag{2}$$

where  $c > 0$ ,  $\alpha > 0$  are certain constants and  $\varepsilon_h \rightarrow 0$ , when  $h \rightarrow 0$ . It should be noted [2] that the last assumption is true for a wide class of continuous cumulative distribution functions in the environment of points whose values are equal to zero. Hence we are allowed to believe that there is an increased possibility of paying out an atypically high compensation together with insurance “duration”. Therefore the formula (2) correctly reflects a phenomenon of a loss and reserve process. Particularly, the equation (2) occurs when  $\Psi(x)$  is a gamma distribution function.

Now we are going to derive certain assessment of a probability distribution of  $X_{Kmin}$  variable together with a risk measure expressed by a probability of reaching a required critical value by values of claims in each of first  $K$  periods. It allows for a simultaneous determination of a probability – in at least one period the value of reserves will not reach such a value. Let us introduce the following symbols:

$$X = X_o + h$$

Hence the formulas (1) and (2) will provide us with the following relation:

$$P(X_{Kmin} \geq x) = [1 - (c + \varepsilon_h) \cdot h^\alpha]^K$$

With

$$h^\alpha = \frac{\delta}{K}$$

where  $\delta > 0$ , as a result for  $K \rightarrow \infty$  we have  $h \rightarrow 0$  and respectively  $\varepsilon \rightarrow 0$ . Then we can write down that

$$P(X_{Kmin} \geq x) = [1 - (c + \varepsilon_h) \cdot \frac{\delta}{K}]^K$$

and moving to the limit for  $K \rightarrow \infty$  we obtain the following relation

$$\lim_{K \rightarrow \infty} P(X_{K \min} \geq x) = e^{-c \cdot \delta}$$

Therefore if we assume that a number of policies or  $K$  periods is high, then as a consequence of

$$\delta = K \cdot (x - X_0)^\alpha$$

we get an approximate formula:

$$P(X_{K \min} \geq x) \approx e^{-c \cdot K \cdot (x - X_0)^\alpha} = l(x)$$

$l(x)$  function determines probability that in the analysed period including  $K$  periods of “functioning” of a given reserve per one portfolio, reserve value is not lower than  $x$ . Values  $c$ ,  $X_0$  and  $\alpha$  are parameters of  $X_{K \min}$  variable distribution. Values to be adopted for a particular set of conditions determining a loss phenomenon in a given insurance risk group may be approximated on the basis of observations [3].

### Example 1

Determine financial “security” of a portfolio in a period  $K = 10000$  periods or in the portfolio consisting of such a number of policies – depending on the values of reserves  $x$  – in order to fulfil the following relation:

$$P(X_{K \min} \geq x) = 0,9999$$

Relation conditioning portfolio “security” shows that a level of risk that is determined by a probability value equalling 0.0001 refers to values of reserves not lower than  $x$  in each  $K$  periods. That is why with reference to the already adopted model assumption there are no conditions that may threat financial security when at least in one analysed period the value of reserves may reach value lower than the assumed one and equalling  $x$ .

To solve this problem we assume that parameters  $c$ ,  $X_0$  and  $\alpha$  have the following values:  $c = 10^{-34}$ ,  $X_0 = 0$  i  $\alpha = 6$ .

According to already adopted symbols in the model the task requires solving of the following equation:  $l(x) = 0.9999$ . To solve this we adopt the following substitution:

$$c \cdot K \cdot x^\alpha = e^{-z}$$

Then

$$l(x) = e^{-e^{-z}}$$

where  $z = -\ln(c \cdot K \cdot x^\alpha) = \alpha \cdot \left[ -\ln x + \frac{1}{\alpha} \cdot \ln \frac{1}{c \cdot K} \right]$

To find a solution effectively a two-stage exponential distribution is applied. In our example the solution is going to be found by substituting  $y = \ln(-\ln e^{-e^{-z}})$  and  $x^* = \ln x$ . The equation will take a form of:

$$y = \alpha \cdot \ln x - \ln \frac{1}{c \cdot K} = \alpha \cdot x^* - \ln \frac{1}{c \cdot K}$$

The equation takes a linear form because of  $x^*$ . On the basis of data in the example of values we obtain the following equation:

$$x = \left( \frac{\ln 0.9999}{-c \cdot K} \right)^{1/\alpha}$$

whose solution reached directly takes the form:  $x = 21544$ . If  $X_0 = 10000$  monetary units the reserves value  $x$  increases and amounts  $x = 22544$ .

Consequently with such data as adopted in the example and the assumed level of risk expressed by a probability value financial security in an insurance company will be sustained because the value of reserves will not be lower than 21,544 monetary units in each analysed period or this value will represent one portfolio policy which may generate any claim as a result of a loss process. This depends on the context of model assumptions. It obviously happens when each individual loss influences a financial result of a portfolio in the same “adverse” way. It also means that such an event like reaching by all variables  $X_i$  a critical value should occur with a determined high probability what suggests that all probability is concentrated on the right side of a distribution.

To provide some comparison – when a level of risk reaches the value of 0.999 the value of reserves should amount to  $x = 31,625$ . Thus, for the value  $x$ , higher probability of sustaining reserves in all three periods lowers i.e. the level of risk goes up. For the value 0.99 value of reserves amounts  $x = 46,454$ , and for the level of risk 0.9 the value of all raised reserves should amount  $x = 68,724$ . However, if a probability equals 0.0001 then  $x = 144.781$ . The above-indicated numerical results should only be treated as an illustration of a model situation that is considered to fix the level of a loss reserve on the basis of information characterised by probability distribution of randomly realised overdue payments that change the level of the reserve randomly.

Basic assumption made in the above considerations concerned lack of any influence of potential conditions that refer to a process of an individual loss occurring per one policy on shaping of this process for another policy. Therefore it was possible to assume some independence of random variables  $X_1, X_2, \dots, X_K$ . This property allows for drawing a conclusion that each policy is somehow an independent element that strongly influences a general financial result of the policy portfolio in case of any loss. A possibility to determine risk associated with compensation payments on a given level is an important tool that facilitates finding out how the financial security of an insurance company may be affected.

## 2. Distribution of claims in the system of $\alpha$ -stable distribution

### 2.1. Assessment of the risk associated with exceeding a critical value by all claims

Let us consider a model situation that refers to exceeding certain value treated as critical by at least one claim. To do this let us consider the event illustrated by the following relation:

$$B^K = \{X^{(1)} < x, \dots, X^K < x\}$$

It is easy to notice that this event probability determines a cumulative distribution function of a random variable:

$$X_{K \max} = \max\{X^{(1)}, \dots, X^{(K)}\}$$

If we assume that random variables  $X^{(i)}$  are of Frecht's distribution. This distribution is  $\alpha$ -stable character determined by the following formula:

$$F(x) = \exp\left\{-\left(\frac{\theta}{x-a}\right)^\nu\right\} \quad \text{dla } x \geq 0$$

and  $F(x) = 0$ , for  $x < 0$ , then depending on the value of  $\nu$  parameter risks of high value claims may be described because moments of this distribution exist only for  $l < \nu$ . The Frecht's distribution is stable as far as maximisation is concerned. Then parameters of  $X^{(i)}$  variables add up and if these variables have identical distribution with identical  $\nu$  and  $\theta$  for all  $i$ , then

$$F_{K \max}(x) = \exp\left\{-K \cdot \left(\frac{\theta}{x-a}\right)^\nu\right\}$$

In this case risk assessment expressed by a given level of a probability of the  $B^K$  event is as follows:

$$1 - P\{B^K\} = 1 - F_{K \max}(x)$$

### Example 1

Determine the risk with given values  $a = 1\,000\,000$ ,  $\nu = 2$ ,  $\theta = 5\,000$  for  $K=10$  and a critical value of claim  $x = 10\,000\,000$ .

In a given case

$$F_{K \max}(x) = \exp\left\{-K \cdot \left(\frac{\theta}{x-a}\right)^\nu\right\} = 0,9999969. \text{ Thus following already adopted}$$

model conditions that determine risk of reaching a critical value by at least one claim we state that this risk measured by a probability cannot exceed 0.000001, and that it is higher for a given critical value and amounts 0.0000031. For  $a = 0$  it amounts 0.0000025. Therefore lowering a level of claims to be considered to zero resulted only in a slight decrease in this risk. This assessment does not meet given standard of security yet. The results undergo a radical change when we lower parameter  $\theta$ . If  $\theta = 500$ , risk level reaches lower than the required value and amounts 0.000 000 025. However with the same value of a parameter



$\theta = 500$  and a number of policies amounting  $K = 1000$  the risk level amounts 0.000 0025. Consequently, the risk radically increased and in this case a situation in the portfolio does not meet given security standards. However, if  $\theta = 50\ 000$  and  $x = 10\ 000\ 000$  the risk level amounts 0,0001 so it is the highest in comparison with all the above presented results.

## 2.2. Application of the Smirnow's distribution

Let us now assume that  $\Psi(x) = 1 - \text{erf}\left(\sqrt{\frac{a}{2 \cdot x}}\right)$  i.e. each claim  $X_i$  has the Smirnow's distribution. Function  $\text{erf}(y)$  is determined by the following formula:

$$\text{erf}(y) = \frac{2}{\sqrt{\pi}} \cdot \int_0^y e^{-u^2} du = \frac{2}{\sqrt{2\pi}} \cdot \int_0^{\sqrt{2}y} \exp\left\{-\frac{t^2}{2}\right\} dt = 2 \cdot \Phi(\sqrt{2} \cdot y)$$

where  $\Phi$  is the Laplace's function (determined by an integral with a lower limit equal 0).

This distribution is  $\alpha$ -stable if  $\alpha = 0,5$ . In this case probability of simultaneous occurrence of the event  $X_i \geq x$  for  $i = 1, \dots, K$  is determined as in the first part by a random variable distribution that is shown by  $X_K$ , in the following way:

$$X_K = \min \{X^{(1)}, \dots, X^{(K)}\}$$

So

$$P\{X^{(1)} \geq x, \dots, X^{(K)} \geq x\} = \left[ \text{erf}\left(\sqrt{\frac{a}{2x}}\right) \right]^K = 2 \cdot \Phi\left(\sqrt{\frac{a}{x}}\right)$$

Matching  $a$  parameter with a predicted scale of a potential level of "realised" reserves or unpaid compensations per a given policy it is possible assess risk regardless what the  $K$  number is. Thus if  $K = 1$  and  $a = 120\ 000$  and  $x = 20\ 000$ , the reserve value on the level not lower than a given value of 20 000 monetary units will be sufficient with a probability of 0,9857. Therefore the insufficiency risk of this reserve is moderate. However, value of reserve realisation on the critical level amounting  $x = 40,000$  is guaranteed with a probability of 0.914. So the risk of such a determined critical value of the reserve is slightly

higher. This risk increases when  $K$  goes up. It is obvious then because it concerns simultaneous reaching of values not lower than  $x$  by all variables  $X^{(i)}$  ( $i = 1, \dots, K$ ).

### 3. Final remarks

The analysis of the risk related with insurance activity carried out in the paper aims at early warning decision-makers of potential threats. Methods of gathering information employed in insurance companies do not meet numerous standards that would guarantee a full implementation of formal techniques that provide decision-makers with reliable knowledge on potential threats. It seems that carrying out scientific research in this field is really useful since in Polish conditions insurance companies operate on the small insurance market and there is no sufficient theoretical and experimental base. Suggested methods are based on the theory of probabilistic and stochastic processes. They may provide a new approach to many problems that bother Polish insurance companies. The problems in question refer both to social and commercial insurance. That is why the suggested approach towards an analytical side of reserves and claims being a launch pad for other analyses in analytic projects may encourage perceiving these issues differently. This way there is much to achieve taking different concepts into consideration while analysing insurance loss phenomena, especially from the perspective of stochastic processes.

In models discussed in the paper the analysis was performed only for the insurance risk in contrary concepts of a risk model and its influence on the general financial equilibrium of the portfolio. Equilibrium was used in its quite colloquial meaning and, therefore, there was not any reference to its proper meaning known from the practice of analyses. To conclude it should be clearly underlined that a distribution of reserves or claims that result in shaking of the insurance company financial equilibrium was determined as a result of reconstructing a loss process in a quite schematic way. For that reason it is only approximately a model of a loss phenomenon characterised by a negative influence of a particular loss on the other loss occurrence. Nevertheless implementation of models being considered here to determine the limit value of reserves and claims that may lead to shaking of the insurance company financial equilibrium certainly results in a more reliable outcome. In case of applying a simple ratio analysis of losses only, it is not possible to draw properly justified conclusions concerning possibilities to determine a moment of shaking of the insurance company finan-

cial equilibrium when conditions that justify implementation of the above-presented model occur.

## References

- Doherty N.A.: Innovations In Managing Catastrophe Risk. "The Journal of Risk and Insurance" 1997, Vol. 64, No. 4.
- Kołmogorow A.N., Gniedenko B.W.: Rozkłady graniczne sum zmiennych losowych niezależnych. PWN, Warszawa 1962.
- Ronka-Chmielowiec W.: Ryzyko w ubezpieczeniach-metody oceny. AE, Wrocław 1997.
- Случайные процессы. Выборочные функции и пересечения. Математика – Новое в зарубежной науке. Ed. J.K. Bielajew. Izdatelstwo „MIR”, Moskwa 1978.
- Szkutnik W.: Wybrane modele w zarządzaniu ryzykiem ubezpieczeniowym w ujęciu probabilistycznym. AE, Katowice 2003.