Uniwersytet Ekonomiczny w Katowicach

Studia Ekonomiczne. Zeszyty Naukowe
Uniwersytetu Ekonomicznego w Katowicach
ISSN 2083-8611 Nr 247 · 2015

Informatyka i Ekonometria 4

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THE APPLICATION OF ALPHA-STABLE DISTRIBUTIONS IN PORTFOLIO SELECTION PROBLEM – THE CASE OF METAL MARKET

Summary: The aim of this article is a brief presentation of the family of alpha-stable distributions and its application in portfolio selection problem. Alpha-stable models are widely used for describing the behaviour of time series observed in financial markets. Leptokurtosis, asymmetry, data clustering and heavy tails in empirical distributions do not allow for inference based on normality approach. These features significantly affect the risk assessment (especially extreme one) and the problem of assets allocation in investment portfolios. The application of alpha-stable models is presented on the example of investment portfolios on metal market.

Keywords: alpha-stable distributions, portfolio selection, risk analysis, metal market.

Introduction

The efficient investment process requires proper assessment of the area where the investment is concerned. Disturbances and fluctuations observed in economy affect significantly investments' decisions. The reflection of economic situation are unpredictable changes of main economic indices, exchange rates and assets prices quoted on capital market. The main goal of any investment is to gain profits. The result of investment undertaken is a certain amount of capital in the future. The most desirable result of any investment is profit, however sometimes the final result may be lower than the invested value. In this case an investor is exposed to the risk of loss. This may happen if individual or portfolio investments are considered.

The classical approach in portfolio theory is based on two characteristics. The first one is expected return measured by expected value of asset's price/return, and the second one is risk measured by standard deviation of asset's price/return. This approach in risk assessment can be applied only in the case if the symmetric distribution, in particular if belongs to the class of elliptical distributions.

The analysis is based on the price returns of financial assets. Therefore in Markowitz portfolio theory the normality assumption is used. From the practical point of view this assumption is not met. The empirical distributions of returns are leptokurtic, asymmetric and heavy-tailed. Taking into account risk analysis these features do not allow for statistical inference based on Gaussian approach. Investors seek to minimize risk for a given level of expected return or to maximize expected return for a given level of risk. Optimization problem requires portfolio's diversification which means that its components should not be correlated. This allows for minimizing risk of the undertaken investment.

1. Alpha-stable distributions

The family of stable distributions was discovered by Paul Lévy [1925] in the second decade of XX, but its connection to financial data was investigated by Mandelbrot [1963] and Fama [1965] in the early sixties. They found that the empirical time series of financial returns were leptokurtic and this discovery forced them to reject normality assumption. As a result they proposed the new class of probability distributions as an alternative to the normal one – stable distributions.

Alpha-stable models are fully described by the four-parameters characteristic function. A random variable $X = \mu + \sigma Z$ belongs to the alfa-stable distribution if, for parameters $\mu \in \mathbb{R}, \sigma > 0$, random variable Z can be described by characteristic function of the form [Samorodnitsky et al., 1994]:

$$\varphi_S(t) = \begin{cases} \exp\left\{-|t|^\alpha \left[1 - i\beta \operatorname{sgn}(t) \tan\frac{\pi\alpha}{2}\right]\right\}, \alpha \neq 1 \\ \exp\left\{-|t| \left[1 + i\beta \frac{2}{\pi} \operatorname{sgn}(t) \ln|t|\right]\right\}, \alpha = 1 \end{cases} \tag{1}$$

where $\alpha \in (0,2]$ is shape parameter (index of stability), $\beta \in [-1,1]$ describes asymmetry, $\mu \in \mathbb{R}$ represents location, $\sigma > 0$ is scale parameter and $\operatorname{sgn}(t)$ refers to signum function. Notation used for alpha-stable distribution is $Stab(\alpha, \beta, \mu, \sigma)$.

The most important parameter is α and describes thickness of the tail of distribution. The smaller values of shape parameter, the heavier tail of the distribution. As mentioned earlier, alpha-stable models are fully described by characteristic function. In theory there exists only four distributions which densities can be represented explicitly by mathematical functions: normal distribution, Cauchy distribution, Lévy distribution and Landau distribution.

The unknown parameters of alpha-stable models are estimated using various of methods. The most popular are Maximum Likelihood Method (ML), Method of Moments (MM) and Quantiles Method (QM) proposed by McCulloch in 1984 [McCulloch, 1984; Krężołek, 2014]. All these methods provide estimates of parameters which are asymptotically normal (under certain assumptions).

2. Alpha-stable portfolios

The construction of investment portfolio is based on a proper allocation of assets. Therefore it affects both the level of risk and expected return. Alpha-stable models are characterized by stability property under the probability summation scheme. It means that the linear combination of independent and identically distributed (iid) random variables with the same index of stability is still alpha-stable random variable with shape parameter α. This property is appropriate only to this class of models. Referring to portfolio theory[1], if random variables $R_1, R_2, \ldots, R_N$ representing returns of N assets in investment portfolio are iid stable random variables with the same shape parameter α: $R_i \sim Stab(\alpha, \beta_i, \mu_i, \sigma_i)$ then the expected portfolio return can be express as [Mittnik et al., 1995]:

$$E[R_P] = \sum_{i=1}^{N} w_i E[R_i] \sim Stab(\alpha, \beta_P, \mu_P, \sigma_P) \tag{2}$$

where:

$$\beta_P = \frac{\operatorname{sgn}(w_1)\beta_1(|w_1|\sigma_1)^\alpha + \cdots + \operatorname{sgn}(w_N)\beta_N(|w_N|\sigma_N)^\alpha}{(|w_1|\sigma_1)^\alpha + \cdots + (|w_N|\sigma_N)^\alpha}, \alpha \in (0,2] \tag{3}$$

$$\sigma_P = [(|w_1|\sigma_1)^\alpha + \cdots + (|w_N|\sigma_N)^\alpha]^{\frac{1}{\alpha}}, \alpha \in (0,2] \tag{4}$$

$$\mu_P = \begin{cases} w_1\mu_1 + \cdots + w_N\mu_N, \alpha \neq 1 \\ -\frac{2}{\pi}(w_1 \ln|w_1|\sigma_1\beta_1 + \cdots + w_N \ln|w_N|\sigma_N\beta_N), \alpha = 1 \end{cases} \tag{5}$$

[1] In case of classical portfolio theory, see Markowitz [1952].

If the shape parameter $\alpha < 2$ the variance of alpha-stable random variable is infinite, thus cannot be considered as a risk measure. Therefore, in terms of investment, expected return can be measured by location parameter and risk can be expressed as a scale parameter od alpha-stable distribution [Rachev et al., 2000; Łażewski et al., 2003].

In N-dimentional case the scale parameter is of the form:

$$\Sigma_P(\mathbf{R}) = [\int_{S_d} |(\mathbf{R}, s)|^{\alpha} \Gamma(ds)]^{\frac{1}{\alpha}} \quad (6)$$

where $\mathbf{R}$ is N-dimensional vector of portfolio components, $S_N \in \mathbb{R}^N, S_d = \{s, \|s\| = 1\}$ in unique sphere in N-dimensional space with finite spectral measure Γ. Thus the optimization problem, which for $\alpha = 2$ represents classical approach proposed by Markowitz, can be solve as:

$$\min_{\mathbf{R} \in \mathbb{R}^N} \Sigma_P(\mathbf{R}) = [\int_{S_N} |(\mathbf{R}, s)|^{\alpha} \Gamma(ds)]^{\frac{1}{\alpha}} \quad (7)$$

The theory of alpha-stable distributions play significant role in construction of investment portfolios. Classical portfolio theory is strongly based on the normality assumption of individual assets. Therefore cannot be used if this assumption is not met. The use of data which not met the assumptions of theoretical models, especially in risk assessment, may generates huge losses. Alpha-stable models are interesting tool for modelling data, extreme risk analysis and portfolio selection problem.

3. Empirical analysis

Metal market is one of the most important part of commodity market and may be considered as an alternative for investors, especially in times of economic crises. In this paper metal market is divided into two sub-markets: precious metals market and non-ferrous metals market. The difference between both is quite clear. Precious metals are widely used in jewellery or medicine while non-ferrous metals are industrial, used in construction sector, aerospace, automotive, etc. Analytically, metal market is not popular field of research and is analysed rather from than investment point of view. It refers to risk assessment as well.

The application of alpha-stable distributions in portfolio theory is presented on the example of investments in metals. The assets considered are: gold, silver, palladium, platinum, copper, aluminium, zinc, thin, lead and nickel. Daily log-returns of spot closing prices are calculated for the period January 2005 – December 2013 (all data quoted on the London Metal Exchange). Referring to the data, if there was no quotation of any asset in some day, this day was removed for all assets considered. The period is divided into two sub-periods:

- sub-period 1 – January 2005 – December 2009: estimation of alpha-stable parameters for all assets and selection of portfolios' components,
- sub-period 2: January 2010 – December 2013: portfolios' analysis.

Figure 1 presents closing prices and log-returns of platinum and copper – all period.

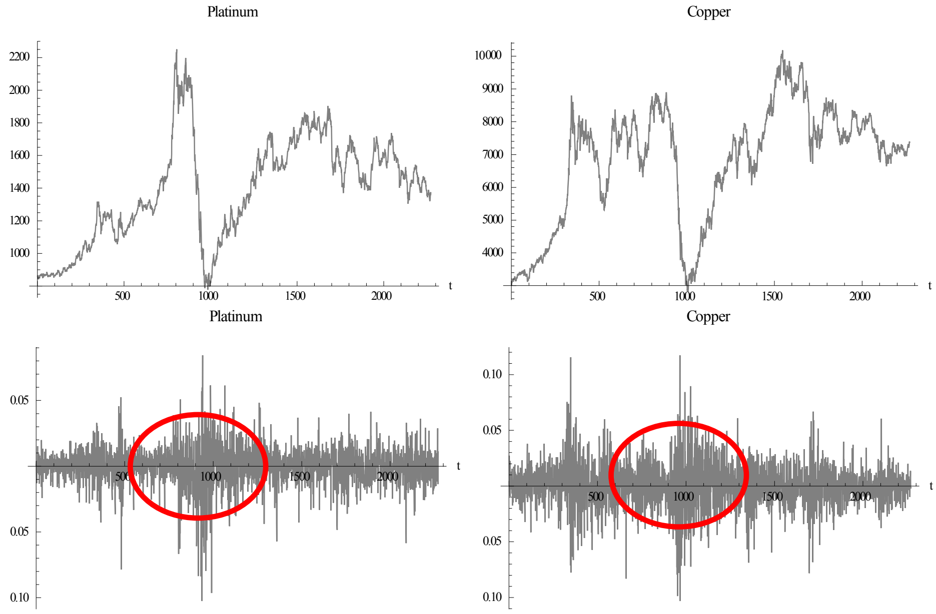

Fig. 1. Closing prices of platinum (top-left) and copper (top-right); log-returns of platinum (bottom-left) and copper (bottom-right)

Source: Own calculations.

Taking into account closing prices, the breakdown on the plot refers to the financial and economic crisis and is clearly reflected in high level of volatility in log-returns (clustering of variance). These features suggest that the normality assumption is supposed to be rejected.

In first stage of analysis the descriptive statistics of all log-returns in sub-period 1 have been calculated. The results are presented in Table 1.

Table 1. Descriptive statistics – sub-period 1

Metal	Mean	Standard deviation	Kurtosis	Skewness	Minimum	Maximum
Gold	0,00075	0,01399	4,66765	-0,15043	-0,07240	0,10245
Silver	0,00077	0,02428	10,05222	-1,27601	-0,20385	0,13180
Palladium	0,00062	0,02262	5,82001	-0,70581	-0,16998	0,09531
Platinum	0,00042	0,01671	5,42955	-0,81259	-0,10259	0,08426
Copper	0,00070	0,02331	2,28210	-0,03781	-0,10321	0,11726
Aluminium	0,00015	0,01717	0,89644	-0,16387	-0,06767	0,06068
Zinc	0,00061	0,02616	1,04311	-0,17402	-0,11472	0,09610
Thin	0,00064	0,02284	4,50803	-0,00249	-0,11453	0,15385
Lead	0,00073	0,02793	1,87539	-0,17770	-0,13199	0,13007
Nickel	0,00021	0,02923	1,81858	0,07291	-0,13744	0,13310

Source: Own calculations.

Descriptive statistics confirm, that empirical distributions of analysed metals are leptokurtic and negative skew, however all generate positive mean. Assuming initially that the log-returns are normally distributed, the estimated values of mean and standard deviation represent unknown parameters of normal distribution.

In the next step is needed to find out if the normality assumption of log-returns is met. Therefore, the goodness-of-fit tests have been used: Anderson--Darling (AD) and Cramer-von Misses tests. The selection of tests is not casual. These tests are commonly used if the empirical distributions supposed to be heavy-tailed. The results are presented in Table 2.

Table 2. Goodness-of-fit tests – sub-period 1 – normal distribution

Metal	AD	p-value	CVM	p-value
	Normal distribution			
Gold	32,74	0,000	5,66	$1,11 \times 10^{-13}$
Silver	37,48	0,000	6,51	$1,44 \times 10^{-15}$
Palladium	29,11	0,000	4,88	$5,62 \times 10^{-12}$
Platinum	45,58	0,000	7,61	0
Copper	39,52	0,000	6,7	$1,22 \times 10^{-15}$
Aluminium	12,45	$1,45 \times 10^{-6}$	1,85	$2,73 \times 10^{-5}$
Zinc	26,49	0,000	4,35	$8,06 \times 10^{-11}$
Thin	42,39	0,000	7,5	0
Lead	28,53	0,000	4,58	$2,55 \times 10^{-11}$
Nickel	33,59	0,000	5,54	$1,99 \times 10^{-13}$

Source: Own calculations.

The results presented in Table 2 suggests that the normality assumption has to be rejected. Therefore, the same tests were used to find out if the distribution was alpha-stable. The parameters of stable models are presented in Table 3 and results of AD and CVM tests are presented in Table 4.

Table 3. Parameters of alpha-stable distribution[2] – sub-period 1

Metal	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\mu}$	$\hat{\sigma}$
Gold	1,73229	-0,25449	0,00061	0,00813
Silver	1,62404	-0,28968	0,00046	0,01248
Palladium	1,61394	-0,00645	0,00108	0,01194
Platinum	1,54764	-0,19900	0,00028	0,00829
Copper	1,74772	-0,08526	0,00061	0,01402
Aluminium	1,87368	-0,29353	0,00006	0,01134
Zinc	1,87777	-0,22823	0,00059	0,01726
Thin	1,55042	-0,17049	0,00009	0,01175
Lead	1,81929	-0,25110	0,00044	0,01759
Nickel	1,80989	0,15947	0,00043	0,01834

Source: Own calculations.

Results presented in Table 3 show that empirical distributions of all metals are heavy-tailed and negative skew. Precious metals have fatter tails comparing to the others, so the probability of huge losses is higher than if use normal approach. Figure 2 shows simulated prices of platinum and zinc compared to the real one within sub-period 2.

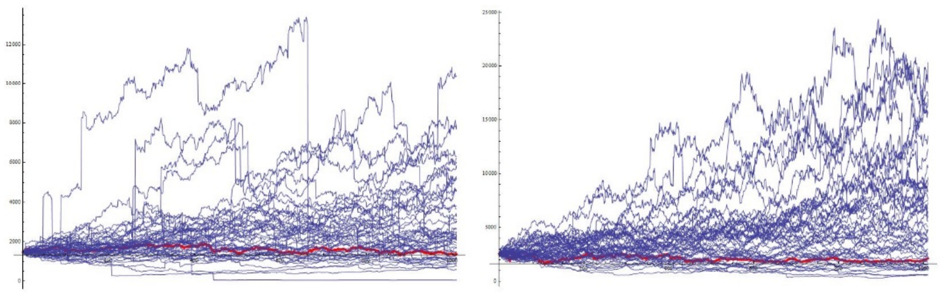

Fig. 2. Simulated prices of platinum (left) and zinc (right) compared to the real one (red line) – subperiod 2

Source: Own calculations.

[2] ML estimates.

Simulated prices confirm high level of volatility, which is covered by alpha-stable models. The goodness-of-fit tests for estimated stable distributions are in Table 4.

Table 4. Goodness-of-fit tests – sub-period 1 – alpha-stable distribution

Metal	AD	p-value	CVM	p-value
	Stable distribution			
Gold	7,66	1,63 x 10^{-4}	1,43	2,53 x 10^{-4}
Silver	1,92	0,102	0,37	0,088
Palladium	1,18	0,175	0,15	0,381
Platinum	3,08	0,025	0,47	0,047
Copper	14,52	3,89 x 10^{-7}	2,43	1,38 x 10^{-6}
Aluminium	5,99	9,91 x 10^{-4}	0,89	0,004
Zinc	16,79	2,76 x 10^{-7}	2,75	2,67 x 10^{-7}
Thin	3,62	0,013	0,59	0,024
Lead	12,08	1,84 x 10^{-6}	1,87	2,55 x 10^{-5}
Nickel	15,07	3,35 x 10^{-7}	2,4	1,61 x 10^{-6}

Source: Own calculations.

As we can find in Table 4, for some metals goodness-of-fit tests do not allow for inferring, that the empirical distribution is alpha-stable. Nevertheless in further analysis this stable models is used. Figure 3 shows empirical and theoretical (alpha-stable) distributions for silver and thin.

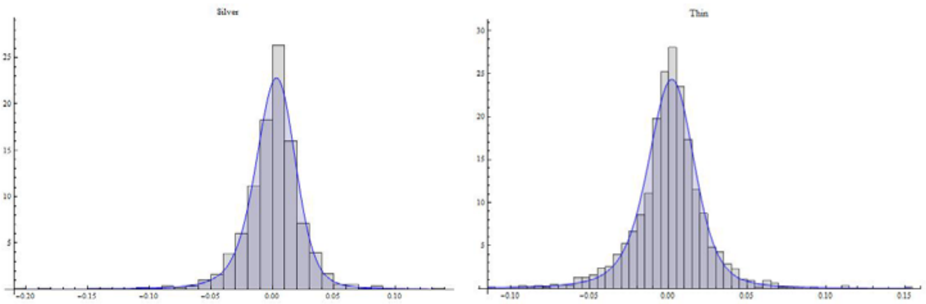

Fig. 3. Empirical and theoretical distribution for silver (left) and thin (right)

Source: Own calculations.

In the next step of analysis, referred to sub-period 2, the components of portfolios have been selected using shape parameter criteria. All assets were ordered using values of alpha. In this paper only two components portfolios are considered. The first portfolio consists of assets with the smallest values of shape parameter in sub-period 1, the second – with the remaining smallest values of alpha, and so on. The final portfolios are presented in Table 5.

Table 5. Portfolios – shape parameter criteria

Portfolio	Components
P1	Platinum/Palladium
P2	Gold/Silver
P3	Thin/Copper
P4	Nickel/Lead
P5	Aluminium/Zinc
P6	Platinum/Thin
P7	Palladium/Silver
P8	Gold/Copper

Source: Own calculations.

Portfolios 1-2 consist of precious metals, portfolios 3-5 consist of non-ferrous metals and portfolios 6-8 consist both of precious and non-ferrous. Figure 4 presents 2-dimentional distributions for portfolios P1 and P4.

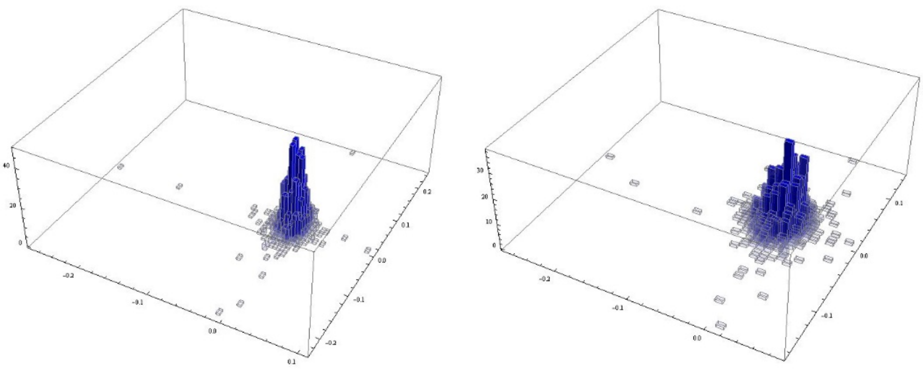

Fig. 4. Empirical and theoretical distribution for silver (left) and thin (right)

Source: Own calculations.

As we can find, there is a lot of points situated in tails of distributions – this confirmed higher probability of huge changes in portfolio returns.

Referring to alpha-stable portfolios, Table 6 shows optimal allocation of metals in each portfolio compared to the classical approach. The optimization is conducted using criteria of minimizing scale parameter. Expected returns and related risks in alpha-stable case are expressed by the location and scale parameters.

Table 6. Optimal portfolios: classical vs. alpha-stable – sub-period 2

Portfolio	Classical approach		Alpha-stable approach		Classical approach		Alpha-stable approach	
	component 1	component 2	component 1	component 2	component 1	component 2	component 1	component 2
P1	70,21%	29,79%	72,19%	27,81%	0,00012*	0,01045	-0,00002	0,00684**
P2	78,44%	21,56%	82,64%	17,36%	0,00010*	0,01030	0,00006	0,00602**
P3	44,00%	56,00%	44,39%	55,61%	0,00012*	0,01192	-0,00006	0,00752**
P4	50,37%	49,63%	48,69%	51,31%	-0,00019*	0,01402	-0,00031	0,00916**
P5	62,19%	37,81%	62,69%	37,31%	-0,00022	0,01124	-0,00019	0,00743**
P6	67,48%	32,52%	66,17%	33,83%	0,00005*	0,01025	-0,00010	0,00662**
P7	57,30%	42,70%	50,15%	49,85%	0,00038*	0,01449	0,00035	0,00908**
P8	65,21%	34,79%	71,97%	28,03%	0,00006*	0,00939	-0,00003	0,00566**

* Higher level of expected return.
** Lower level of risk.

Source: Own calculations.

When comment the results, although in classical approach the level of expected return in almost each portfolios is higher comparing to alpha-stable models, the risk related to all investments is lower for alpha-stable portfolios (Figure 5). Moreover, the weights for all components in portfolios differ in both cases. This conclusion is significant in decision making process.

Fig. 5. Expected returns vs. risk for alpha-stable (PS) and normal (PN) allocation

Source: Own calculations.

Summary and conclusions

The family of alpha-stable distributions is commonly used in models, where the normality assumption is rejected. Returns observed in financial markets are exposed for unexpected changes caused not only by market factors. In this paper that class of models is applied to portfolio theory. The area of research is metal market – an interesting alternative for investors especially when financial or economic crises occur. As presented, the use of alpha-stable models in portfolio's construction allows for reducing level of risk, which is significant in decision making process.

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ZASTOSOWANIE ROZKŁADÓW ALFA-STABILNYCH W ZAGADNIENIU BUDOWY PORTFELA INWESTYCYJNEGO – PRZYPADEK RYNKU METALI

Streszczenie: Celem artykułu jest zwięzła prezentacja rozkładów alfa-stabilnych oraz ich zastosowanie w teorii portfela inwestycyjnego. Modele alfa-stabilne są powszechnie wykorzystywane w naukach ekonomiczno-finansowych do opisu rozkładów prawdopodobieństwa danych przedstawionych w postaci szeregów czasowych. Empiryczne stopy zwrotu obserwowane na rynku cechuje wysoki poziom leptokurtozy, asymetrii (często lewostronnej), zjawisko skupiania zmienności oraz grube ogony empirycznych rozkładów stóp zwrotu. Cechy te uniemożliwiają prowadzenie wnioskowania statystycznego bazującego na paradygmacie normalności. Ponadto rozkłady alfa-stabilne są ściśle związane z zagadnieniem wyboru modelu opisującego ryzyko, zwłaszcza ekstremalne, oraz z zagadnieniem budowy portfela inwestycyjnego. Klasyczna teoria Markowitza, wobec niespełnienia założenia o normalności rozkładu, może być stosowana, jednakże z dużą dozą ostrożności. Odpowiednia alokacja składników w portfelu jest determinowana przyjętym rozkładem probabilistycznym, a tym samym wpływa na podejmowanie decyzji inwestycyjnych. Zastosowanie rozkładów alfa-stabilnych przedstawiono na przykładzie inwestycji na rynku metali.

Słowa kluczowe: rozkłady alfa-stabilne, budowa portfela inwestycyjnego, analiza ryzyka, rynek metali.