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## MODELING EXTREME MORTALITY RISK

**Summary:** The main aim of the paper is presentation some key aspects in modeling extreme mortality risk. We make a review and discuss measures of extreme mortality risk. Besides, we use approach proposed by J.M. Bravo et al. [2012], that is focused on using EVT to model the statistical behaviour of mortality rates over a given high threshold age. Insurers and reinsurers are interested in assessing the risk exposure to extreme mortality risk.

**Keywords:** extreme risk, mortality, extreme value theory.

### Introduction

Over the last century, the annual number of deaths all over the world has in general decreased. As of 2014 the crude death rate for the whole world was 7.89 per 1000 (down from 8.37 per 1000 in 2009) [CIA World Factbook, 2016]. Mortality improvement is a clear evidence of how far society and science have come in improving general living conditions, promoting healthier lifestyles, offering better medical and healthcare services, that extend our lives [Bravo et al., 2012].

While mortality has been improving for many decades, life insurers still have to deal with the risk, that such event as a pandemic or terrorist attack could cause a one-time mortality shock. It is not easy to determine in advance the loss value from such events, and the amount of capital to hold. This makes it difficult for life insurers to quantify the risk and to manage their capital efficiently for the benefit of policyholders and shareholders.

Measuring, modeling and managing mortality risk (and as consequence – longevity risk) is a huge challenge for risk managers. The financial effect of underestimating the life table limiting age can be substantial, not only in terms

of expected losses, but particularly in terms of risk measures such as VaR or TailVaR since these quantities heavily rely on the tail of the population survival distribution.

In this paper we consider some aspects of extreme mortality risks.

The natural tool for modeling extreme mortality risk is extreme value theory (EVT), that provides a framework to formalize the analysis of the behavior in the tails of a distribution. In the first part, we discuss some of the most popular risk measures for mortality rates. In the second part, we focus on using EVT to model the behavior of mortality rates over a given high threshold level of age.

## 1. (Extreme) mortality risk

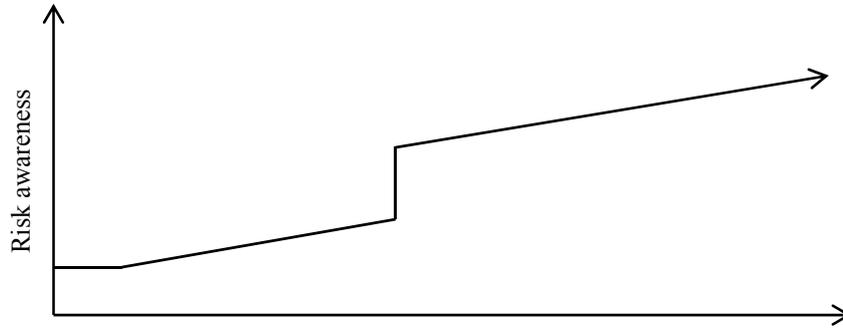
Let's consider the underlying aggregate mortality rate  $q(t,x)$  in year  $t$  at age  $x$ , which is usually unobservable. What is observed depends on the way of recording deaths and population size by national statistical offices [Cairns et al., 2008]. Usually, we observe the crude death rate  $m_c(t,x)$  ( $c$  symbolizes crude death rate) calculated as: the number of deaths,  $D(t,x)$ , aged  $x$  last birthday at the date of death during year  $t$ , divided by the exposure,  $E(t,x)$  which is the average population aged  $x$  last birthday during year  $t$ . The crude death rate is the simplest and one of the most frequently used mortality indicators in population [Cairns et al., 2008]. Generally, the term "mortality risk" is used to cover all forms of deviations in aggregate mortality rates from those anticipated at different ages and over different time horizons [Cairns et al., 2008].

A key driver of capital requirements for annuity business is the uncertainty over the trend of future mortality rates. The uncertainty in future death rates can be divided into two components [Cairns et al., 2008]:

- unsystematic mortality risk: even if the true mortality rate is known, the number of deaths,  $D(t,x)$ , will be random; the larger is the population, the smaller is the unsystematic mortality risk (as a result of the pooling of offsetting risks, i.e. diversification),
- systematic mortality risk: this is the undiversifiable component of mortality risk, that affects all individuals in the same way; specifically, forecasts of mortality rates in future year are uncertain.

So far, high foreseeable mortality rates were interrupted by HIV/AIDS epidemic or event of September 11, 2001, that brought additional attention to the risk of mortality jumps. Risk of such dramatic shocks in mortality rates caused by terrorist attack or pandemic has been increased and awareness of the existence of this risk has been growing (Fig. 1). As a consequence, we have to deal with

short-term, catastrophic, extreme mortality risk, which is referred to the risk that, over short periods of time, mortality rates are very much higher than would normally be experienced [Cairns et al., 2008].



HIV epidemic Sept 11, 2001 SARS scare Swine flu epidemic  
1990 Deaths from AIDS terrorist attack Bird flu pandemic risk

**Fig. 1.** Timeline of the growing awareness of the extreme mortality risk

Source: [A. Krutov, 2010, s. 238].

## 2. Extreme mortality risk measurement

The goal of risk measurement, in general, is to use an appropriate risk measure to assign a real number to an uncertainty or a quantity with an unknown value, so that the risk exposure of this quantity can be represented. Value at Risk and Tail Value at Risk are used usually for measuring mortality risk, especially extreme risk (more about extreme risk in study [Trzpiot, 2015]).

VaR is one of the most widely used risk measures by insurers and banks in quantitative risk management. The VaR is the maximum loss not exceeded with a given level of confidence over a given time horizon. Formally, we write  $VaR_\alpha(x) = \inf\{x \mid F_x(x) \geq \alpha\}$ , where  $F_x(x)$  is the distribution function of  $x$  and  $\alpha$  is the tolerance level  $\alpha \in (0,1)$ . A loss distribution has to be specified first and the VaR is a quantile of that distribution. VaR has become the benchmark risk measure. The VaR does also not consider the shape of the tail beyond the confidence level. Besides, VaR is not sub-additive, which makes it not very suitable to use for a capital requirement calculation.

Tail VaR (TVaR) overcomes this problem. TVaR is the expected loss conditional on the loss exceeding VaR. If VaR is determined using a confidence

level of  $\alpha$ , then tail VaR is the expected loss over the other  $1 - \alpha$  part of the loss distribution. Formally, TVaR is defined as:  $TVaR_\alpha(x) = \frac{1}{1 - \alpha} \int_\alpha^1 VaR_\lambda(x) d\lambda$

Tail VaR satisfies all coherence requirements.

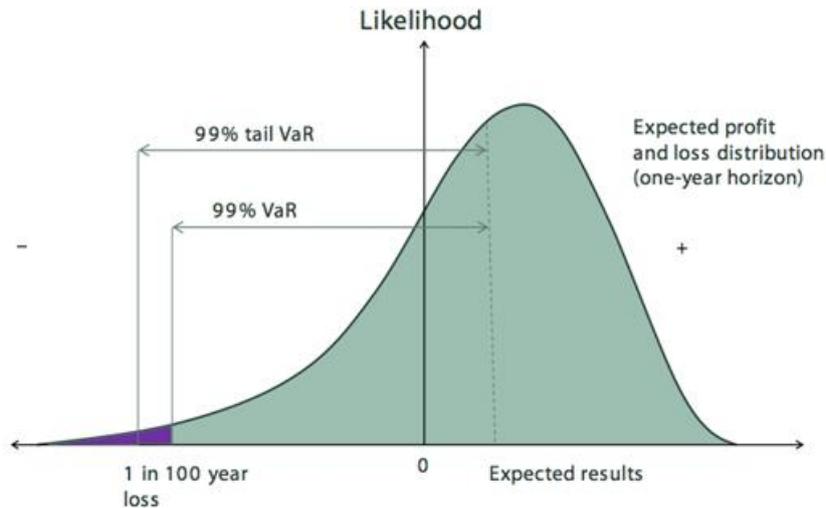
VaR is the risk measure of Solvency II<sup>1</sup> with a confidence level of 99,5% for the calculation of the Solvency Capital Requirement and 85% for the Minimum Capital Requirement. In the case of the SCR, the confidence level of 99,5% tells us, that the company can expect to lose no more than VaR in the next year with 99,5% confidence, so, on average, only once every 200 years the VaR loss level will be exceeded. Assuming VaR equals 100, the probability for capital to become negative (ruin) in the  $x$  following years will amount to  $(1 - \alpha)\%$ , provided the company holds 100 of initial capital [CEIOPS, 2006].

TVaR is the risk measure used for the Swiss Solvency Test (SST)<sup>2</sup> with a confidence level of 99%. Assuming TVaR equals 115, it means that in the worst cases (the  $(1 - \alpha)\%$  situations in which capital becomes negative in the  $x$  following years), the company will lose 115 in the average. If the company holds 115 of capital, it should therefore survive these worst-case scenarios (very roughly half of the time [CEIOPS, 2006]).

The figure 2 illustrates the difference between the VaR risk measure (used by Solvency II) and the Tail VaR risk measures (used by SST).

<sup>1</sup> Solvency II is a new supervisory framework, which is in force from 2016 for insurers and reinsurers in Europe. It puts demands on the required economic capital, risk management and reporting standards of insurance companies. Solvency II focuses on an enterprise risk management approach towards required capital standards. Its main objective is to ensure, that insurance companies hold sufficient economic capital to protect the policyholder as it aims to reduce the risk, that an insurance company is unable to meet its financial claims. The capital is adjusted to the risks, that the insurance company incurs. In this way, Solvency II describes the quantitative requirements. This capital requirement is called the Solvency Capital Requirement (SCR) and covers all the risks, that an insurer faces.

<sup>2</sup> The Swiss Solvency Test (SST) is a risk based capital standard for insurance companies in Switzerland in use since 2006. The SST was developed by the Swiss Federal Office of Private Insurance in cooperation with the Swiss insurance industry.



**Fig. 2.** Comparison of VaR and Tail VaR

Source: [M. Eves, 2013, s. 3].

In 2006 the European Insurance and Occupational Pensions Authority (EIOPA)<sup>3</sup> generally acknowledged the theoretical advantages of using the TVaR to calculate the SCR. However, there were concerns about the utility of an SCR measurement based on TVaR, since there was seen to be scarcity of data about the tails, which can easily lead to an increase in modeling errors. J. Yamai and T. Yoshida [2004] argue, that the estimation errors of TVaR are much greater than those of VaR. This is a potential drawback of the TVaR. These estimation errors can however be reduced by increasing the sample size of the simulation or by making assumptions on the shape of the tail. Another disadvantage is that the TVaR does not exist for distributions with an infinite mean, these distributions are however seldom associated with profit loss distributions and therefore this disadvantage is not relevant in this case.

The SCR is defined as the 99,5% VaR of the Available Capital over one year. The Available Capital at time  $t$ ,  $AC_t$ , is the difference between market value of assets and market value of liabilities at time  $t$ . Thus, it is a measure of the amount of capital that is available to cover future losses, i.e. the smallest amount  $x$  for which [Bauer et al., 2009]:

<sup>3</sup> EIOPA's core responsibilities are to support the stability of the financial system, transparency of markets and financial products as well as the protection of insurance policyholders, pension scheme members and beneficiaries.

$$P(AC_1 > 0 \mid AC_0 = x) \geq 99,5\%$$

However, since this implicit definition is rather unpractical in numerical computations, one often uses the following approximately equal definition [Bauer et al., 2009]:  $SCR^{VaR} := \operatorname{argmin}_x \left\{ P \left( AC_0 - \frac{AC_1}{1+i_1} > x \right) \leq 0,005 \right\}$

In general, we can write SCR definition as:

$$\rho(AC_{t+1} - AC_t \mid F_t)$$

where  $F_t$  denotes a filtration representing today's information and  $\rho$  denotes a risk measure (VaR at 99,5%, Tail VaR at 99%).

Mortality trend risk in then the one-year view of the SCR definition is the potential deviation of next year's best estimates from today's best estimates. In other words, mortality trend risk can be defined as the risk of unexpected changes in the (long-term) trend underlying the future mortality evolution [Börger et al., 2013]. This risk is relevant for insurance products, which pay upon death of the insured person as well as products, which pay upon survival.

A comprehensive risk model for mortality (or longevity) meeting the Solvency II and SST criteria requires an adequate stochastic mortality model. In particular the one-year time horizon and focus on extreme mortality trend deviations deserve special attention. Moreover, an adequate stochastic model needs to be efficient and sufficiently simple to maintain. A wide range of different models has been proposed in the literature (an overview of different types of models see: [Cairns et al., 2008]). But only very few of the existing models are directly applicable for the calculation of capital requirements under Solvency II or the SST.

### 3. Modeling extreme mortality risk using extreme value theory

Traditional actuarial approaches to modeling mortality risk are not usable in the context of analyzing extreme mortality. These approaches have been developed to use historical data for quantifying stable mortality rates and identifying and incorporating in the rates the trends of slowly shrinking mortality. A standard mortality table is of little use in trying to quantify the risk of a sudden jump in mortality rates due to an event such as pandemic or large-scale terrorist attack.

A challenge in mortality risk modeling is the knowledge about rare mortality events. The statistical methods for evaluating extreme events require an accurate measure of the tail of the distribution.

Extreme value theory (EVT) can be used not only to model the given sample of observations in the tail, but also to extrapolate the probability of even more extreme, out-of-sample events. Broadly speaking, there are two types of models for extreme values. Block maxima models apply to maxima of a sequence of observations and the Peaks-Over-Threshold models deal with exceedances over a given high threshold<sup>4</sup>. In case of mortality rates we are rather interested in the exceedances in the tail of the distribution, because the payoff of the mortality indexed bond occurs, when mortality exceeds or falls below a certain level.

Suppose we have a sequence of i.i.d. random variables  $X_1, \dots, X_n$ , representing risks or losses, from an unknown common distribution function  $F$  and let  $M_n = \max \{X_1, \dots, X_n\}$ . A natural measure of extreme events are the values of  $X_i$ , that exceed a high threshold  $u$ . Let  $x_0$  be the finite or infinite right endpoint of the distribution:

$$F: x_0 = \sup \{x \in R : F(x) < 1\} \leq \infty$$

We define the excess distribution above the threshold  $u$  as the conditional probability:

$$F_u(x) = P\{X - u \leq x \mid X > u\} = \frac{F(x+u) - F(u)}{1 - F(u)} \quad (1)$$

for  $0 \leq x < x_0 - u$ .

$F_u(x)$  is the probability that  $X$  exceeds the threshold  $u$  by no more than an amount  $x$ , given that the threshold is exceeded.

According to the Pickand-Balkema-de Haan Theorem [Balkema, de Haan, 1974], for a sufficiently high threshold  $u$ , the excess distribution function  $F_u(x)$  may be approximated by the generalized Pareto distribution (GPD),  $G_{\xi, \theta}(x)$ , for some value of  $\xi$  and  $\theta$ . The GPD is defined as:

$$G_{\xi, \theta}(x) = \begin{cases} 1 - \left(1 + \frac{\xi x}{\theta}\right)^{-1/\xi} & \text{if } \xi \neq 0 \\ 1 - \exp\left(-\frac{x}{\theta}\right), & \text{if } \xi = 0 \end{cases} \quad (2)$$

where  $\theta > 0$ , and the support is  $x \geq 0$  when  $\xi \geq 0$  and  $0 \leq x \leq -\theta/\xi$  when  $\xi < 0$ .  $\xi$  represents the shape parameter of the distribution or tail index and  $\theta$  is an addi-

<sup>4</sup> For a detailed review of this subject see: [e.g. Embrechts, Klüppelberg, Mikosch, 2008].

tional scaling parameter. For  $\xi > 0$  we get a reparameterized version of the ordinary Pareto distribution. The case  $\xi = 0$  corresponds to the exponential distribution and  $\xi < 0$  is known as a type II Pareto distribution.

There is possibility to add location parameter  $\gamma$  to the GPD family. The GPD  $G_{\xi,\gamma,\theta}(x)$  is then defined to be  $G_{\xi,\theta}(x-\gamma)$ . Therefore, for  $x \geq u$ , the distribution function of the exceedances  $F_u(x-u)$  may be approximated by  $G_{\xi,\theta}(x-u) = G_{\xi,u,\theta}(x)$ .

When we focus on the behaviour of mortality rates over a given high threshold age  $u$  then  $G_{\xi,u,\theta}(x)$  can be interpreted as below (the approach was proposed by [Bravo et. al., 2012]).

Let  $X$  represent the time-to-death random variable for a person aged 0. Then, for some high age  $u$ ,  $G_{\xi,u,\theta}(x)$  represents the probability, that the person will die before age  $u+x$ , given survival to age  $u$ . The GPD distribution provides us with closed expressions for yearly death probabilities  $q_x$  and mortality forces  $\mu_x$ . For a given high age  $x \geq u$ , we can derive:

$$q_x = 1 - \left( 1 + \frac{\xi}{\theta + \xi(x-u)} \right)^{-1/\xi} \quad (3)$$

and

$$\mu_x = \frac{1}{\theta + \xi(x-u)} \quad (4)$$

As J.M. Bravo et al. [2012] underlined the above results have been given in terms of stationary sequences of random variables. But they can be adapted for use with data from non-stationary sequences, in which characteristics of the stochastic process change with modifications in some related random variable. For instance, the distribution of life spans might shift upward/downward over time due to medical breakthroughs, pandemic episodes or cohort specific covariates. In this case, the GPD parameters can be expressed as functions of time and that information can be used to project the evolution of extreme life spans over time.

The one of the biggest problem in application of EVT is the choice of the threshold above, which observations are accepted as extremes. In this case it is the choice of the threshold age  $u$ . The choice of the threshold age  $u$  is called in the literature a trade-off between the bias and the variance of parameter estimates. On the one hand, we need to choose a high enough  $u$ , so that the GPD can be applied asymptotically to mortality data (which means reducing bias). Too high threshold means not enough observations to obtain efficient estimates. On the

other hand, if  $u$  is determined too low many ordinary data are taken as extreme ones, thus yielding biased estimates. In both cases, the resulting estimates may lead to misleading conclusions when assessing risk. To identify the optimal threshold value different methods are used in the literature:

- graphical tools, namely to an empirical mean excess function plot or to a plot of the index maximum likelihood estimators resulting from using increasing thresholds,
- common sense-based choices of the cut-off (e.g. choose  $u$  in such a way, that about 5%-15% of the data are thought of as extreme observations),
- Monte Carlo simulation methods,
- algorithms (based for instance on the bootstrap method), that endogenously determine the cut-off, the most suited to the data.

The threshold life table model developed by S.H. Li et al. [2008] addresses this issue by adopting a piecewise approach in which the threshold age is chosen in a statistical way without the need of any subjective decision and in which the fitted statistical distribution determines, if exists, the appropriate end point of the life table. It means, that in a threshold life table death rates at earlier adult ages are graduated by means of a parametric function (the classical B. Gompertz [1825] mortality law), and at advanced ages the threshold life table model assumes a given extreme value statistical distribution, here the GPD [Bravo et al., 2012].

Let  $z = \{X - u | X > u\}$  be the conditional exceedances of the age at death  $X$  over a given threshold age  $u$ . Based on the EVT the threshold life table model is defined as [Bravo et al., 2012; Li et al., 2008]:

$$F(x) = \begin{cases} 1 - \exp\left(-\frac{B}{\ln C}(C^x - 1)\right) & \text{if } x \leq u \\ 1 - \left(1 + \frac{\xi(x-u)}{\theta}\right)^{-1/\xi} & \text{if } x > u \end{cases} \quad (5)$$

In conclusion, model (5) assumes, that the survival distribution is Gompertzian before the threshold age, and the exceedances over the threshold age  $u$  follow a GPD. To ensure that  $F(x)$  is a proper distribution function, the following constraints must be satisfied:  $B > 0$ ,  $C > 1$  and  $\theta < 0$ .

The model (5) guarantees, that  $F(x)$  is continuous at the threshold age  $u$ , but that the smoothness of  $F(x)$  around  $u$  is not guaranteed what makes this step very sensitive part in empirical applications. However, parametric or non-parametric graduation methods may be needed to smooth the mortality curve around the threshold age  $u$ .

S.H. Li et al. [2008] proposed two methods for choosing the threshold age: a maximum likelihood estimation method and a weighted least-squares estimation method. Maximum likelihood estimation method is described below (details of this method and weighted least-squares estimation method are described in details in [Li et al., 2008]).

Let  $l_x$  denote the number of survivors to age  $x$ , the number of deaths between ages  $x$  and  $x + 1$  is  $d_x = l_x - l_{x+1}$ . The likelihood contribution for each age  $x = x_{\min}, x_{\min} + 1, \dots, x_{\max} - 1$  is the probability of dying between age  $x$  and age  $x + 1$ , raised to the number of deaths or:

$$\left( \frac{s(x) - s(x+1)}{s(x_{\min})} \right)^{d_x}$$

where  $s(x) = 1 - F(x)$  is the survival function in (5). The likelihood contribution for the survivors to age  $x_{\max}$  is the probability of survival to age  $x_{\max}$ , raised to the number of survivors or:

$$\left( \frac{s(x_{\max})}{s(x_{\min})} \right)^{l_{x_{\max}}}$$

The resulting likelihood function is described as:

$$L(B, C, \xi, \theta, u) = \left[ \prod_{x=x_{\min}}^{x_{\max}-1} \left( \frac{s(x) - s(x+1)}{s(x_{\min})} \right)^{d_x} \right] \times \left( \frac{s(x_{\max})}{s(x_{\min})} \right)^{l_{x_{\max}}} \quad (6)$$

The logarithm of  $L(B, C, \xi, \theta, u)$  can be decomposed into the sum of two components,  $l_1(B, C, u) + l_2(\xi, \theta, u)$ , where:

$$l_1(B, C, u) = \sum_{x=x_{\min}}^{u-1} d_x \ln(s(x) - s(x+1)) + l_u \ln(s(u)) - l_{x_{\min}} \ln(s(x_{\min}))$$

and

$$l_2(\xi, \theta, u) = \sum_{x=u}^{x_{\max}} d_x \left( \frac{s(x)}{s(u)} - \frac{s(x+1)}{s(u)} \right) + l_{x_{\max}} \ln\left( \frac{s(x_{\max})}{s(u)} \right)$$

where:

$$\frac{s(x)}{s(u)} = \left(1 + \frac{\xi(x-u)}{\theta}\right)^{-1/\xi}$$

For a fixed age  $u$ , parameter estimation for the parametric [Gompertz, 1825] modelling part and the generalized Pareto part can be done separately by maximizing  $l_1$  and  $l_2$ , respectively. The choice of  $u$  depends on the maximization of the profile log-likelihood function  $l_p$ :

$$l_p(u) = l(\hat{B}(u), \hat{C}(u), \hat{\xi}(u), \hat{\theta}(u), u)$$

where  $l = \ln(L)$ ,  $\hat{B}(u)$ ,  $\hat{C}(u)$ ,  $\hat{\xi}(u)$ ,  $\hat{\theta}(u)$  are the maximum likelihood estimates of  $B$ ,  $C$ ,  $\xi$  and  $\theta$  for a fixed  $u$ , respectively.

The process of threshold age and model parameter estimation can be summarized by the following algorithm (example for age  $u = 98$ ) [Li et al., 2008]:

Step 1:

- A. Find the values of  $B$  and  $C$  that maximize  $l_1$ .
- B. Find the values of  $\xi$  and  $\theta$  that maximize  $l_2$ .
- C. Compute the value of the profile likelihood  $l_p$ .

Step 2: Repeat Step 1 for  $u = 97, 96, \dots, 85$ .

Step 3: Find the value of  $u$  that gives the maximum profile log-likelihood.

The value of  $u$  obtained in step 3 is the optimal threshold age. The maximum likelihood estimates of  $B$ ,  $C$ ,  $\xi$ , and  $\theta$  under the optimal threshold age are the ultimate model parameter estimates.

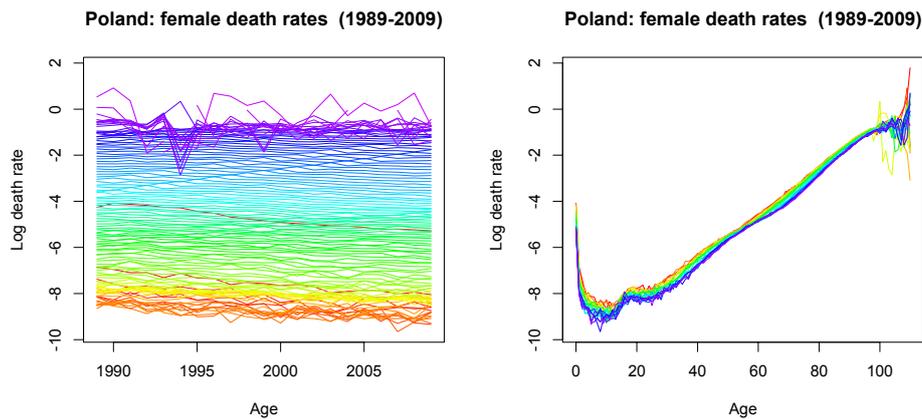
In the threshold life table model, the form of the tail is determined by the model parameters, and hence, the data. If  $\xi < 0$ , then  $q_x = 1$  when to  $\omega = u - \theta/\xi$ ; if  $\xi > 0$ , then  $q_x$  tends asymptotically to 1, and that if  $\xi = 0$ , then the tail is exponential, implying that  $q_x$  tends to a limit that is less than 1.

#### 4. Illustration of modeling extreme mortality risk using EVT

We use the threshold life table to model mortality rates for the total population of Poland. The data are period life table functions (deaths, number of survivors etc.) by single years of age up to 110+ provided by the Human Mortality Database [2016]. All calculations are made in R environment.

In Figure 3 we take a look at the evolution of death rates by age and year (1989-2009) in the overall population of Poland. On the left years are plotted using a rainbow palette, so the earliest years are red, followed by orange, yellow, green, blue and indigo with the most recent years plotted in violet. On the right each age is shown as a separate time series in a time plot.

We observe, that in the last thirty years the death rates have been declining steadily at all ages, with greater speed at ages between 65 and 85.



**Fig. 3.** Death rates  $q_x$  by year and age for Poland (total population)

Source: Calculation in R, package demography.

Table 1 exhibits the parameter estimates and corresponding standard errors for the Gompertz and GPD components of the optimal threshold life table, the optimal threshold age and the mean excess life time at the optimal threshold age in 2009.

**Table 1.** Parameter estimates, standard errors and p-values for the Gompertz and GPD parameters, optimal threshold age and mean excess life time (total Polish population, for year 2009)

Parameter estimates	Male	Female	Total
$\ln B$ (SE)	-12735 (0,135)	-11836 (0,128)	-14398 (0,139)
p-value	< 0,0001	< 0,0001	< 0,0001
$\ln C$ (SE)	0,1375 (0,001)	0,1163 (0,001)	0,1273 (0,001)
p-value	< 0,0001	< 0,0001	< 0,0001
$\xi$ (SE)	-0,1635 (0,002)	-0,1735 (0,001)	-0,1563 (0,001)
p-value	< 0,0001	< 0,0001	< 0,0001
$\theta$ (SE)	3,783 (0,067)	3,534 (0,046)	3,429 (0,063)
p-value	< 0,0001	< 0,0001	< 0,0001
$u$	93	96	94
$e(u)$	< 0,0001	< 0,0001	< 0,0001

Source: Own calculations in R.

We observe that the parameter estimates of the Gompertz mortality law and of the GPD function are all statistically significant at the optimal threshold age.

The same empirical analysis was done by J.M. Bravo et al. [2012]. They used mortality data for the total, male and female populations of Portugal, Spain and France. They also observe a good fit of the model in all populations and subperiods analysed and on the whole life span considered.

## Conclusions

This paper addresses the problem of extreme mortality risk – which is referred to the risk that, over short periods of time, mortality rates are very much higher than would normally be experienced – and discusses the ways in which life insurers, annuity providers and pension plans can model this kind of risk.

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#### MODELOWANIE EKSTREMALNEGO RYZYKA UMIERALNOŚCI

**Streszczenie:** Celem niniejszej pracy jest przedstawienie kluczowych aspektów w modelowaniu ekstremalnego ryzyka umieralności. Przedstawiamy dwa podejścia pomiaru ryzyka. Po pierwsze, dyskutujemy miary ryzyka ekstremalnego, które są wykorzystywane w pomiarze ryzyka umieralności. Po drugie, przedstawiamy podejście zaproponowane przez J.M. Bravo i innych [2012], polegające na wykorzystaniu EVT do modelowania umieralności powyżej pewnego wieku. Oceną ekstremalnego ryzyka umieralności są zainteresowani ubezpieczyciele i reasekuratorzy.

**Słowa kluczowe:** ekstremalne ryzyko, umieralność, teoria wartości ekstremalnych.